COMPOSITIO MATHEMATICA

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Compositio Math. **146** (2010), 561–598.

doi:10.1112/S0010437X09004448

FOUNDATION COMPOSITIO MATHEMATICA

The London Mathematical Society

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The semisimplicity conjecture for A-motives

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Abstract

We prove the semisimplicity conjecture for A-motives over finitely generated fields K. This conjecture states that the rational Tate modules $V_{\mathfrak{p}}(M)$ of a semisimple A-motive M are semisimple as representations of the absolute Galois group of K. This theorem is in analogy with known results for abelian varieties and Drinfeld modules, and has been sketched previously by Tamagawa. We deduce two consequences of the theorem for the algebraic monodromy groups $G_{\mathfrak{p}}(M)$ associated to an A-motive M by Tannakian duality. The first requires no semisimplicity condition on M and states that $G_{\mathfrak{p}}(M)$ may be identified naturally with the Zariski closure of the image of the absolute Galois group of K in the automorphism group of $V_{\mathfrak{p}}(M)$. The second states that the connected component of $G_{\mathfrak{p}}(M)$ is reductive if M is semisimple and has a separable endomorphism algebra.

Contents

1. Introduction

The aim of this article is to prove the following result, which is called the *semisimplicity conjecture for A-motives*. We use standard notation and terminology (A, F, their completions $A_{\mathfrak{p}}, F_{\mathfrak{p}}, \ldots$), which are introduced formally in § 2. The uninitiated reader may think of the case of t-motives, where $A = \mathbb{F}_q[t]$, $F = \mathbb{F}_q(t)$ and, in the case of $\mathfrak{p} = (t)$, we have $A_{\mathfrak{p}} = \mathbb{F}_q[\![t]\!]$ and $F_{\mathfrak{p}} = \mathbb{F}_q(\!(t)\!)$.

Theorem 1.1. *Let K be a field which is finitely generated over a finite field. Let M be a semisimple A-motive over K of characteristic ι. Let $\mathfrak{p} \neq \ker \iota$ be a maximal ideal of A. Then the rational Tate module $\mathrm{V}_{\mathfrak{p}}(M)$ associated to M is semisimple as a $\mathfrak{p}$-adic representation of the absolute Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ of K.*

Received 5 September 2008, accepted in final form 7 June 2009, published online 18 March 2010.
2000 Mathematics Subject Classification 11G09 (primary), 11F80, 13A35 (secondary).
Keywords: A-motive, t-motive, semisimple, Galois representation, monodromy group, semilinear algebra.

The strategy of our proof of the semisimplicity conjecture is not original, it has been sketched by Tamagawa [Tam95].

Using the categorical machinery of the author's previous article [Sta08], the following consequences for the algebraic monodromy groups of A-motives ensue formally from Theorem 1.1.

Theorem 1.2. *Let K be a field which is finitely generated over a finite field. Let M be an A-motive over K of characteristic ι, not necessarily semisimple. Let $\mathfrak{p} \neq \ker \iota$ be a maximal ideal of A. Let $G_{\mathfrak{p}}(M)$ be the algebraic monodromy group of M, and let $\Gamma_{\mathfrak{p}}(M)$ denote the image of the absolute Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ of K in $\mathrm{Aut}_{F_{\mathfrak{p}}}(\mathrm{V}_{\mathfrak{p}}(M))$.*

(a) *The natural inclusion $\Gamma_{\mathfrak{p}}(M) \subset G_{\mathfrak{p}}(M)(F_{\mathfrak{p}})$ has Zariski-dense image.*

(b) *If M is semisimple and its endomorphism algebra is separable, then the connected component of $G_{\mathfrak{p}}(M)$ is a reductive group.*

The concept of effective A-motives was invented by Anderson [And86] in the case $A = \mathbb{F}_q[t]$ for perfect K under the name of t-motives. They may be viewed as analogues of Grothendieck's pure motives, and even the conjectural heart of Voevodsky's derived mixed motives, with the essential difference that both the field of definition *and* the ring of coefficients of an A-motive are of positive characteristic. For an introduction to the theory of A-motives we refer to the original source [And86] and the books of Goss [Gos96] and Thakur [Tha04].

The semisimplicity conjecture is an analogue of the Grothendieck–Serre conjecture which asserts the semisimplicity of the étale cohomology groups of pure motives. This analogue has been proven only in the case of abelian varieties, by Faltings [Fal83] for fields of definition of characteristic zero, and by Zarhin [Zar76] for fields of definition of positive characteristic.

The semisimplicity conjecture is closely connected with two other conjectures, the Tate conjecture and the isogeny conjecture. Only the conjunction of the Tate conjecture with the semisimplicity conjecture allows us to deduce the consequences for the algebraic monodromy groups of A-motives. The Tate conjecture characterises Galois-invariant endomorphisms of the associated Tate modules. It has been proven independently by Tamagawa [Tam94a] and Taguchi [Tag95, Tag96] and will be reproven in this article (Proposition 5.16). The isogeny conjecture on the other hand is a fundamental finiteness statement which, as in the case of abelian varieties, implies both the Tate conjecture and the semisimplicity conjecture. For fields of definition of transcendence degree at most one, the isogeny conjecture has been proven quite recently by Pink [Pin08], using a different method. It seems that his results combined with ours allow us to deduce the isogeny conjecture for all finitely generated fields of definition.

A special class of A-motives arises from Drinfeld modules. All such A-motives are semisimple, and the semsimplicity conjecture for this class has been proven previously by Taguchi in [Tag91, Tag93] for fields of definition of transcendence degree at most one, using a different strategy inspired by [Fal83]. Pink and Traulsen have extended this proof to direct sums of Drinfeld modules in [PT06].

We end the introduction with an overview of this article. In § 2 we construct the rigid tensor category $A\text{-}\mathrm{Mot}_K$ of A-motives in the spirit of Taelman [Tae09], containing the full subcategory $A\text{-}\mathrm{Mot}_K^{\mathrm{eff}}$ of effective A-motives. Inverting isogenies, we obtain the Tannakian category of A-isomotives. We introduce the integral Tate module functors $\mathrm{T}_{\mathfrak{p}}$ with values in the categories of integral $\mathfrak{p}$-adic Galois representations $\mathrm{Rep}_{A_{\mathfrak{p}}}(\Gamma_K)$. They induce the rational Tate module functors $\mathrm{V}_{\mathfrak{p}}$ with values in the Tannakian categories of rational $\mathfrak{p}$-adic Galois representations $\mathrm{Rep}_{F_{\mathfrak p}}(\Gamma_K)$.

$$\begin{tikzcd}
A\text{-}\mathrm{Mot}_K^{\mathrm{eff}} \arrow[r, phantom, "\subset"] & A\text{-}\mathrm{Mot}_K \arrow[r, "\mathrm{T}_{\mathfrak p}"] \arrow[d] & \mathrm{Rep}_{A_{\mathfrak p}}(\Gamma_K) \arrow[d, "F_{\mathfrak p}\otimes_{A_{\mathfrak p}}(-)"] \\
& A\text{-}\mathrm{Isomot}_K \arrow[r, "\mathrm{V}_{\mathfrak p}"'] & \mathrm{Rep}_{F_{\mathfrak p}}(\Gamma_K)
\end{tikzcd}$$

Section 3 begins with the introduction of some terminology for semilinear algebra: the notions of bold rings $\boldsymbol{R}$, bold modules $\boldsymbol{M}$, restricted bold modules and bold scalar extension of modules from one bold ring to another. Its main result, Theorem 3.11, concerns the two fundamental properties of bold scalar extension in a special situation.

In § 4 we show that the category of A-isomotives embeds into the category $\boldsymbol{F_K}\text{-}\mathrm{Mod}^{\mathfrak{p}\text{-res}}$ of $\mathfrak{p}$-restricted bold modules over a certain bold ring $\boldsymbol{F_K}$. We recall the classification of $\mathfrak{p}$-adic Galois representations in terms of the category $\boldsymbol{F_{K,\mathfrak p}}\text{-}\mathrm{Mod}^{\mathfrak{p}\text{-res}}$ of $\mathfrak{p}$-restricted $\boldsymbol{F_{K,\mathfrak p}}$-modules, which employs the functor $\mathrm{D}_{\mathfrak p}$ of Dieudonné modules. In this translation to semilinear algebra, the functor induced by the Tate module functor is of a rather simple form, it is the functor $\boldsymbol{F_{K,\mathfrak p}} \otimes_{\boldsymbol{F_K}} (-)$ of bold scalar extension from $\boldsymbol{F_K}$ to $\boldsymbol{F_{K,\mathfrak p}}$. Following Tamagawa, we introduce an intermediate bold ring $\boldsymbol{F_K} \subset \boldsymbol{F_{\mathfrak p,K}} \subset \boldsymbol{F_{K,\mathfrak p}}$, which allows us to factor the above bold scalar extension functor through the category of $\boldsymbol{F_{\mathfrak p,K}}\text{-}\mathrm{Mod}^{\mathfrak{p}\text{-res}}$ of $\mathfrak{p}$-restricted $\boldsymbol{F_{\mathfrak p,K}}$-modules.

$$\begin{tikzcd}
A\text{-}\mathrm{Isomot}_K \arrow[rr, "\mathrm{V}_{\mathfrak p}"] \arrow[d, hook, "I"'] & & \mathrm{Rep}_{F_{\mathfrak p}}(\Gamma_K) \arrow[d, "\mathrm{D}_{\mathfrak p}"', "\cong"] \\
\boldsymbol{F_K}\text{-}\mathrm{Mod}^{\mathfrak{p}\text{-res}} \arrow[r, "\boldsymbol{F_{\mathfrak p,K}}\otimes_{\boldsymbol{F_K}}(-)"'] & \boldsymbol{F_{\mathfrak p,K}}\text{-}\mathrm{Mod}^{\mathfrak{p}\text{-res}} \arrow[r, "\boldsymbol{F_{K,\mathfrak p}}\otimes_{\boldsymbol{F_{\mathfrak p,K}}}(-)"'] & \boldsymbol{F_{K,\mathfrak p}}\text{-}\mathrm{Mod}^{\mathfrak{p}\text{-res}}
\end{tikzcd}$$

The main result of § 3 then implies that the bold scalar extension functor $\boldsymbol{F_{\mathfrak p,K}} \otimes_{\boldsymbol{F_K}} (-)$ maps semisimple objects to semisimple objects.

Sections 5 and 6 follow Tamagawa in constructing a certain bold ring $\boldsymbol{B}$ which induces a functor $\mathrm{C}_{\mathfrak p}$ from rational $\mathfrak{p}$-adic Galois representations to $\mathfrak{p}$-restricted $\boldsymbol{F_{\mathfrak p,K}}$-modules. All of this is very much in the spirit of Fontaine theory, note however that we are dealing with global Galois representations, not local Galois representations as in Fontaine theory.

$$\begin{tikzcd}
& \mathrm{Rep}_{F_{\mathfrak p}}(\Gamma_K) \arrow[dl, "\mathrm{C}_{\mathfrak p}"'] \arrow[d, "\mathrm{D}_{\mathfrak p}"] \\
\boldsymbol{F_{\mathfrak p,K}}\text{-}\mathrm{Mod}^{\mathfrak{p}\text{-res}} \arrow[r, "\boldsymbol{F_{K,\mathfrak p}}\otimes_{\boldsymbol{F_{\mathfrak p,K}}}(-)"'] & \boldsymbol{F_{K,\mathfrak p}}\text{-}\mathrm{Mod}^{\mathfrak{p}\text{-res}}
\end{tikzcd}$$

The functor $\mathrm{C}_{\mathfrak p}$ has a variety of favourable properties. Among others, it allows us to decide which Galois representations arise from a $\mathfrak{p}$-restricted $\boldsymbol{F_{\mathfrak p,K}}$-module[1] by a numerical criterion. It also ensures that the bold scalar extension functor $\boldsymbol{F_{K,\mathfrak p}} \otimes_{\boldsymbol{F_{\mathfrak p,K}}} (-)$ maps semisimple objects to semisimple objects. Thereby, the proof of Theorem 1.1 is completed.

Finally, § 7 introduces the algebraic monodromy groups associated to A-isomotives via Tannakian duality applied to the fibre functor $\mathrm{V}_{\mathfrak p}$ of Tate modules. We deduce Theorem 1.2 from Theorem 1.1, using results from my article [Sta08].

[1] Tamagawa calls such representations quasigeometric.

2. *A*-Isomotives

Let F be a global field of positive characteristic p, with finite field of constants $\mathbb{F}_q$ of cardinality q. Fix a finite non-empty set $\{\infty_1, \ldots, \infty_s\}$ of places of F, the 'infinite' places. Denote by A the subring of F consisting of those elements integral outside the infinite places. Choose a field K containing $\mathbb{F}_q$, and set $A_K := A \otimes_{\mathbb{F}_q} K$, this is a Dedekind ring. Choose also an $\mathbb{F}_q$-algebra homomorphism $\iota\colon A \to K$, it corresponds to a prime ideal $\mathfrak{P}_0$ of A_K of degree one. If ι is injective, we say that the characteristic is generic. If not, we say that the characteristic is special.

Let σ_q denote the Frobenius endomorphism $c \mapsto c^q$ of K, and let σ denote the induced endomorphism $a \otimes c \mapsto a \otimes c^q$ of A_K. For any A_K-module M, a *σ-linear map* $\tau\colon M \to M$ is an additive map which satisfies $\tau(r \cdot m) = \sigma(r) \cdot \tau(m)$ for all $(r, m) \in A_K \times M$.

Note that to give a σ-linear map $\tau\colon M \to M$ is equivalent to giving its *linearisation*

$$\tau_{\mathrm{lin}}\colon \sigma_* M := A_K \otimes_{\sigma, A_K} M \to M, \quad r \otimes m \mapsto r \cdot \tau(m),$$

which is an A_K-linear map.

Definition 2.1. An *effective A-motive over K* (of characteristic ι) is a finitely generated projective A_K-module M together with a σ-linear map $\tau\colon M \to M$ such that the support of $M/(A_K \cdot \tau M)$ is contained in $\{\mathfrak{P}_0\}$. The *rank* $\mathrm{rk}(M)$ of an effective A-motive (M, τ) is the rank of its underlying A_K-module M.

Definition 2.2. Let M and N be effective A-motives over K. A *homomorphism* $M \to N$ is an A_K-linear map that commutes with τ. An *isogeny* is an injective homomorphism with torsion cokernel (as a homomorphism of A_K-modules).

The category $A\text{-}\mathrm{Mot}_K^{\mathrm{eff}}$ of effective A-motives over K is an A-linear category. While the kernels and cokernels of all homomorphisms exist categorically, it is not an abelian category since the categorical kernel and cokernel of an isogeny are both zero, even though not all isogenies are isomorphisms. Note that, conversely, a homomorphism of effective A-motives with zero categorical kernel and cokernel is an isogeny.

Definition 2.3. Let (M, τ_M) and (N, τ_N) be effective A-motives over K. The *tensor product* $M \otimes N$ of M and N is the effective A-motive consisting of the A_K-module $M \otimes_{A_K} N$ together with the σ-linear map

$$\tau\colon M \otimes_{A_K} N \to M \otimes_{A_K} N, \quad m \otimes n \mapsto \tau_M(m) \otimes \tau_N(n).$$

Endowed with this tensor product, the category $A\text{-}\mathrm{Mot}_K^{\mathrm{eff}}$ is an associative, commutative and unital tensor category. The unit $\mathbf{1}$ is given by A_K itself, equipped with the σ-linear map σ itself. However, it is not a rigid tensor category, since the dual of an effective A-motive M does not exist except if its τ_{lin} is bijective.

Proposition 2.4. *Let L, M, N be effective A-motives over K. If L is of rank one, then the natural homomorphism*

$$\mathrm{Hom}(M, N) \longrightarrow \mathrm{Hom}(M \otimes L, N \otimes L), \quad f \mapsto f \otimes \mathrm{id}$$

is an isomorphism.

Remark 2.5. If a dual $L^\vee$ of L would exist in the category of effective A-motives, then Proposition 2.4 would be trivial: we could simply 'twist back' using $L^\vee$. This is true more generally for invertible objects in tensor categories, and we use this fact in the following without further mention.

Proof [2]. The given homomorphism is induced by the bijective homomorphism

$$\mathrm{Hom}_{A_K}(M, N) \to \mathrm{Hom}_{A_K}(M \otimes_{A_K} L, N \otimes_{A_K} L),\ f \mapsto f \otimes \mathrm{id}$$

of the underlying A_K-modules, so it is injective. An A_K-linear map $g = f \otimes 1 : M \otimes_{A_K} N \to M \otimes_{A_K} N$ is a homomorphism of effective A-motives if $(f \circ \tau_M) \otimes \tau_L = (\tau_N \circ f) \otimes \tau_L$. This implies that $f \circ \tau_M = \tau_N \circ f$, so f is a homomorphism of effective A-motives, as required. □

DEFINITION 2.6. An *A-motive over K* is a pair $X = (M, L)$ consisting of two effective A-motives over K of which L is of rank one.

DEFINITION 2.7. Let (M', L') and (M, L) be A-motives over K. A *homomorphism* $(M', L') \to (M, L)$ of A-motives is a homomorphism $M' \otimes L \to M \otimes L'$ of effective A-motives over K. If the latter is an isogeny, then we say that the given homomorphism of A-motives is an isogeny.

Example 2.8. Let $X = (M, L)$ be an A-motive. For every $0 \neq a \in A$, the homomorphism $M \otimes_{A_K} L \to M \otimes_{A_K} L$, $m \otimes l \mapsto a \cdot m \otimes l$ is an isogeny $[a]_X : X \to X$, the *scalar isogeny* of X induced by a.

Given this definition of homomorphisms of A-motives, it is not completely obvious how to compose two homomorphisms. We will use Proposition 2.4. Let $X' = (M', L')$, $X = (M, L)$ and $X'' = (M'', L'')$ be A-motives over K. We define the composition of homomorphisms as follows, where the isomorphisms are given by Proposition 2.4 and $\to$ is the composition of homomorphisms of effective A-motives.

$$\begin{array}{c}
\mathrm{Hom}(X', X) \times \mathrm{Hom}(X, X'') \\
\Big\| \\
\mathrm{Hom}(M' \otimes L, M \otimes L') \times \mathrm{Hom}(M \otimes L'', M'' \otimes L) \\
\Big\downarrow \cong \\
\mathrm{Hom}(M' \otimes L \otimes L'', M \otimes L' \otimes L'') \times \mathrm{Hom}(M \otimes L' \otimes L'', M'' \otimes L' \otimes L) \\
\Big\downarrow \\
\mathrm{Hom}(M' \otimes L \otimes L'', M'' \otimes L' \otimes L) \\
\Big\uparrow \cong \\
\mathrm{Hom}(M' \otimes L'', M'' \otimes L') \\
\Big\| \\
\mathrm{Hom}(X', X'')
\end{array}$$

The category $A\text{-}\mathrm{Mot}_K$ of A-motives over K is an A-linear category. Note that the direct sum of two A-motives $X' = (M', L')$ and $X = (M, L)$ is given by $X' \oplus X = ((M' \otimes L) \oplus (M \otimes L'), L' \otimes L)$.

We have a natural functor from effective A-motives to A-motives, mapping M to $(M, \mathbf{1})$.

[2] Compare [Tae09, Lemma 2.3.1].

Definition 2.9. The *tensor product* of two A-motives $X' = (M', L')$ and $X = (M, L)$ is the A-motive

$$X' \otimes X = (M' \otimes M, L' \otimes L).$$

Definition 2.10. Let $X = (M, L)$ be an A-motive, and let $d \geqslant 0$ be an integer. The *dth exterior power* $\bigwedge^d X$ of X is the A-motive $(\bigwedge^d M, L)$, where $\bigwedge^d M$ denotes the dth exterior power of the A_K-module underlying M together with the unique σ-linear endomorphism such that the homomorphism $\bigotimes_{A_K}^d M \to \bigwedge_{A_K} M$ is a homomorphism of A-motives.

We denote the second-highest and highest non-trivial exterior powers of X as $M^* := \bigwedge^{\mathrm{rk}(M)-1} M$ and $\det(M) := \bigwedge^{\mathrm{rk}(M)} M$, respectively.

Proposition 2.11. *The category A-Mot_K of A-motives over K is a rigid A-linear tensor category, and the natural functor A-$\mathrm{Mot}_K^{\mathrm{eff}} \to A$-$\mathrm{Mot}_K$ is a fully faithful A-linear tensor functor.*

Proof. We suppress the details, remarking only that the dual of an A-motive $X = (M, L)$ is given by $X^\vee := (M^* \otimes L, \det M)$. $\square$

Considering A-$\mathrm{Mot}_K^{\mathrm{eff}}$ as a subcategory of A-Mot_K, we note that an A-motive $X = (M, L)$ is the internal Hom $\mathcal{H}om(L, M)$ of the effective A-motives M and L.

The category of A-motives is again not an abelian category. To obtain such a category, we must invert those homomorphisms which have both zero kernel and zero cokernel in the categorical sense, the isogenies. We start by studying isogenies more carefully.

We will see that every isogeny is a factor of a scalar isogeny (Proposition 2.20). This will allow us to 'invert isogenies' by inverting scalar isogenies, technically a simpler task.

Definition 2.12.

(a) A *torsion $\boldsymbol{A_K}$-module* is a finitely generated torsion A_K-module T together with a σ-linear map $\tau : T \to T$. A homomorphism of torsion $\boldsymbol{A_K}$-modules is a τ-equivariant homomorphism of A_K-modules. The category of torsion $\boldsymbol{A_K}$-modules is an A-linear abelian category, and has an evident tensor product.

(b) We say that a torsion $\boldsymbol{A_K}$-module (T, τ) is *of characteristic ι* if the supports of both kernel and cokernel of τ_{lin} are contained in $\{\mathfrak{P}_0\}$.

Given an isogeny $f : M \to N$ of effective A-motives, the quotient $T := N/f(M)$ in the category of A_K-modules inherits a σ-linear map, so T is a torsion $\boldsymbol{A_K}$-module. Note that it is of characteristic ι. If necessary, we denote (T, τ) by $\mathrm{coker}_{\boldsymbol{A_K}}(f)$.

Definition 2.13. Let $f : M' \to M$ be an isogeny of effective A-motives, and set $(T, \tau) := \mathrm{coker}_{\boldsymbol{A_K}}(f)$. The isogeny f is *separable* if τ_{lin} is bijective. The isogeny is *purely inseparable* if τ is nilpotent. We extend these two notions to isogenies of A-motives via the corresponding isogenies of effective A-motives.

With an eye towards our interest in isogenies of A-motives, we turn to a discussion (Theorem 2.17) of the structure of the associated torsion $\boldsymbol{A_K}$-modules of characteristic ι.

We intersperse a discussion of the connection of torsion $\boldsymbol{A_K}$-modules with bijective τ_{lin} with Galois representations. The natural place for this would be later in the article, but it will be useful in the proof of the next theorem.

DEFINITION 2.14. Let $\Gamma_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$ denote the absolute Galois group of K. An *A-torsion Galois representation* is an A-module V of finite length together with a group homomorphism $\rho\colon \Gamma_K \to \mathrm{Aut}_A(V)$.

DEFINITION 2.15.

(a) Let (T, τ) be a torsion $\boldsymbol{A_K}$-module such that τ_{lin} is bijective. We set $R_q(T, \tau) := (K^{\mathrm{sep}} \otimes_K T)^\tau$, taking τ-invariants with respect to the diagonal action.[3] Note that the action of Γ_K on K^{sep} induces an action of Γ_K on $R_q(T, \tau)$.

(b) Let (V, ρ) be an A-torsion Galois representation. We set $D_q(V, \rho) := (K^{\mathrm{sep}} \otimes_{\mathbb{F}_q} V)^{\Gamma_K}$, taking Γ_K-invariants with respect to the diagonal action.[4] Note that the σ-linear endomorphism σ_q of K^{sep} induces a σ-linear endomorphism τ of $D_q(V, \rho)$.

PROPOSITION 2.16. *Let $\Gamma_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$ denote the absolute Galois group of K. The functors D_q, R_q are quasi-inverse equivalences of A-linear rigid abelian tensor categories.*

$$\left(\!\!\left(\begin{matrix} A\text{-torsion} \\ \text{Galois representations}\end{matrix}\right)\!\!\right) \underset{R_q}{\overset{D_q}{\rightleftarrows}}\!\!{\scriptstyle\cong}\ \left(\!\!\left(\begin{matrix} \text{torsion } \boldsymbol{A_K}\text{-modules} \\ \text{with bijective } \tau_{\mathrm{lin}}\end{matrix}\right)\!\!\right)$$

Moreover, the following is true:

(a) $\dim_K D(V, \rho) = \dim_{\mathbb{F}_q} V$ *for every A-torsion Galois representation;*

(b) *the homomorphism $K^{\mathrm{sep}} \otimes_{\mathbb{F}_q} R_q(T, \tau) \to K^{\mathrm{sep}} \otimes_K T$ is an isomorphism for every torsion $\boldsymbol{A_K}$-module (T, τ) with bijective τ_{lin};*

(c) *the homomorphism $K^{\mathrm{sep}} \otimes_K D_q(V, \rho) \to K^{\mathrm{sep}} \otimes_{\mathbb{F}_q} V$ is an isomorphism for every A-torsion Galois representation (V, ρ).*

Proof. Forgetting the A-module structure of both sides, this is Proposition 4.1 of [PT06] and its proof. By naturality of that proposition, the statement of our proposition holds. □

THEOREM 2.17. *Let (T, τ) be a torsion $\boldsymbol{A_K}$-module of characteristic ι.*

(a) *If $\ker \iota = 0$, then τ_{lin} is bijective.*

(b) *If $\ker \iota \neq 0$, then there exists a canonical filtration*

$$0 \to (T', \tau') \to (T, \tau) \to (T'', \tau'') \to 0$$

of (T, τ) by torsion $\boldsymbol{A_K}$-modules such that τ'_{lin} is bijective and τ'' is nilpotent.

(c) *If τ is nilpotent, then there exists a canonical filtration of (T, τ) by torsion $\boldsymbol{A_K}$-modules such that each successive subquotient is annihilated by τ.*

(d) *We have $\mathrm{Ann}_A(T) \neq 0$.*

Proof [5].

(a) Since $\mathfrak{P}_0$ lies over the generic prime of A, we have:

$$\text{The prime ideals } \sigma_*^m(\mathfrak{P}_0) \text{ for } m \geqslant 0 \text{ are pairwise different.} \tag{2.18}$$

[3] We use 'R' for representation.
[4] We use 'D' for Dieudonné.
[5] The author is grateful to Gebhard Böckle for helping to simplify this proof.

Set $X := \ker(\tau_{\mathrm{lin}})$ and $Y := \operatorname{coker}(\tau_{\mathrm{lin}})$. We consider the exact sequence of A_K-modules

$$0 \longrightarrow X \longrightarrow \sigma_* T \xrightarrow{\tau_{\mathrm{lin}}} T \longrightarrow Y \longrightarrow 0.$$

To every finitely generated torsion A_K-module $N \cong \bigoplus_{\mathfrak{a}} A_K/\mathfrak{a}$ we may associate its characteristic ideal $\chi(N) := \prod \mathfrak{a}$. We have $\dim_K X = \dim_K Y$, so $\chi(X) = \chi(Y) = \mathfrak{P}_0^n$ for some $n \geqslant 0$, and

$$\chi(\sigma_* T) = \chi(T). \tag{2.19}$$

Now (2.19) means that σ_* permutes the (finitely many) prime ideals lying in the support of T. Therefore, for every such prime ideal $\mathfrak{P}$ in the support there exists an integer $m \geqslant 0$ such that $\sigma_*^m \mathfrak{P} = \mathfrak{P}$. Now (2.18) excludes the possibility that $\mathfrak{P}_0$ is contained in the support of T. It follows that both X and Y are zero, so τ_{lin} is indeed bijective.

(b) Note that $\operatorname{im}(\tau_{\mathrm{lin}}^m) = A_K \cdot \tau^m(T)$. Since T has finite length, this chain of submodules becomes stationary and $T' := \bigcap_{m \geqslant 0} \operatorname{im}(\tau_{\mathrm{lin}}^m) = \operatorname{im}(\tau_{\mathrm{lin}}^n)$ for some $n \gg 0$. In particular, the restriction of τ_{lin} to T' is bijective, and the induced σ-linear endomorphism of $T'' := T/T'$ is nilpotent.

(c) Clearly, $\tau_{\mathrm{lin}}(T) \subset T$ is a τ-invariant A_K-submodule. The induced action of τ on the quotient $T/\tau_{\mathrm{lin}}(T)$ is zero by construction. Since T has finite length, we may repeat this construction to obtain a filtration with the desired properties.

(d) It is sufficient to prove the statement for the successive subquotients of any chosen filtration of (T, τ) by torsion $\boldsymbol{A_K}$-modules. We use those given by items (b) and (c).

If τ_{lin} is bijective, then the A-torsion Galois representation associated by Proposition 2.16 has finite length as A-module, so it has non-zero annihilator in A. Again by Proposition 2.16, it follows that T itself has non-zero annihilator in A.

If τ is zero and T is non-zero, then $T = \operatorname{coker} \tau_{\mathrm{lin}}$ has support contained in $\{\mathfrak{P}_0\}$. By part (a) we have $\mathfrak{P}_0 \cap A = \ker \iota \neq 0$, so again T has non-zero annihilator in A.

Using parts (a), (b), (c) and the previous special cases, it follows that $\operatorname{Ann}_A(T) \neq 0$ for all torsion $\boldsymbol{A_K}$-modules (T, τ) of characteristic ι. $\square$

Proposition 2.20. *Every isogeny is a factor of a scalar isogeny. More precisely, let $f : X' \to X$ be an isogeny of A-motives over K. There exists an element $0 \neq a \in A$, and an isogeny $g : X \to X'$ such that $g \circ f = [a]_{X'}$ and $f \circ g = [a]_X$, so the following diagram commutes.*

$$\begin{tikzcd}
& X \arrow[rr, "{[a]_X}"] \arrow[dr, "g"] & & X \\
X' \arrow[ur, "f"] \arrow[rr, "{[a]_{X'}}"'] & & X' \arrow[ur, "f"'] &
\end{tikzcd}$$

In particular, the relation of isogeny is an equivalence relation.

Proof. We may assume that both X' and X are effective A-motives. Let $(T, \tau) := \operatorname{coker}_{\boldsymbol{A_K}}(f)$, a torsion $\boldsymbol{A_K}$-module of characteristic ι. By Theorem 2.17(d), there exists an element $0 \neq a \in A$ such that $a \cdot T = 0$. Therefore, $a \cdot X$ is contained in $f(X') \cong X$, so we obtain an isogeny $X \xrightarrow{g} X'$ with $f \circ g = [a]_X$. Since f is a homomorphism of $\boldsymbol{A_K}$-modules, we have

$$f \circ g \circ f = [a]_X \circ f = f \circ [a]_{X'},$$

so since f is injective we obtain $g \circ f = [a]_{X'}$. $\square$

We include the following consequence of Theorem 2.17, it will not be needed in the following.

PROPOSITION 2.21. *Let $X' \xrightarrow{f} X''$ be an isogeny of A-motives.*

(a) *If $\ker \iota = 0$, then f is separable.*

(b) *If $\ker \iota \neq 0$, then there exist canonically an A-motive X and a factorisation $f = f'' \circ f'$,*

$$\begin{tikzcd} X' \arrow[rr, "f"] \arrow[dr, "f'"'] & & X'' \\ & X \arrow[ur, "f''"'] & \end{tikzcd}$$

such that $f' : X' \to X$ is a separable isogeny and $f'' : X \to X''$ is a purely inseparable isogeny.

Proof. (a) We may assume that all A-motives involved are effective. Set $(T, \tau) := \operatorname{coker}_{\boldsymbol{A_K}}(f)$. If $\ker \iota = 0$, then τ_{lin} is bijective by Theorem 2.17(a), so f is separable.

(b) If $\ker \iota \neq 0$, Theorem 2.17(b) gives us a canonical filtration

$$0 \to (T', \tau') \to (T, \tau) \to (T'', \tau'') \to 0$$

such that τ'_{lin} is bijective and τ'' is nilpotent. Letting X be the inverse image of T' in X'', we obtain an effective A-motive such that f factors as desired. □

DEFINITION 2.22. An *A-isomotive over K* is an A-motive over K. A homomorphism of A-isomotives is an F-linear combination of homomorphisms of A-motives. More precisely, given two A-isomotives X', X, we set

$$\operatorname{Hom}_{A\text{-Isomot}_K}(X', X) := F \otimes_A \operatorname{Hom}_{A\text{-Mot}_K}(X', X),$$

where A-Isomot$_K$ denotes the category of A-isomotives over K.

We might say that an A-isomotive is *effective* if it is isomorphic in A-Isomot$_K$ to an effective A-motive. We remark that some authors use the terminology *F-motive* for what we call an A-isomotive in this article.

THEOREM 2.23.

(a) *The natural functor A-Mot$_K \to A$-Isomot$_K$ is universal among A-linear functors with target an F-linear category and mapping isogenies to isomorphisms.*

(b) *The category A-Isomot$_K$ is an F-linear rigid abelian tensor category.*

Proof.

(a) Our given functor is A-linear by definition. It maps isogenies to isomorphisms by Proposition 2.20. Let $\mathcal{C}$ be an F-linear category, and let $V : A\text{-Mot}_K \to \mathcal{C}$ be an A-linear functor which maps isogenies to isomorphisms.
It remains to show that there exists a unique A-linear functor $V' : A\text{-Isomot}_K \to \mathcal{C}$ extending V. Since A-Mot$_K$ and A-Isomot$_K$ have the same objects, we turn our attention to homomorphisms. Since scalar isogenies *are* isogenies, and V *does* map isogenies to isomorphisms, the desired extension V' exists and is unique.

(b) The category of A-isomotives is F-linear by construction. It inherits a rigid tensor product from the category of A-motives. We must show that it is abelian. For this, assume that $f : X' \to X$ is a homomorphism of A-isomotives with vanishing categorical kernel and cokernel. We may assume that both X' and X are effective A-isomotives. By the definition of homomorphisms of A-isomotives, there exists an element $0 \neq a \in A$ such that $a \cdot f\colon X' \to X$ is a homomorphism of effective A-motives. The categorical kernel and cokernel of $a \cdot f$ remain zero, since multiplication by a is an isomorphism. Clearly, this implies that $a \cdot f$ is injective, and $\operatorname{coker}_{\boldsymbol{A_K}}(a \cdot f)$ is a torsion $\boldsymbol{A_K}$-module. Therefore, $a \cdot f$ is an isogeny, and Proposition 2.20 gives an element $0 \neq b \in A$ and an isogeny $g\colon X \to X$ such that $(a \cdot f) \circ g$ and $g \circ (a \cdot f)$ are both multiplication by b. Since multiplication by b is an isomorphism in $A\text{-Isomot}_K$, this implies that f is an isomorphism. □

Definition 2.24. An A-motive M is *semisimple* if it is such as an object of the category of A-isomotives.

We turn to $\mathfrak{p}$-adic Galois representations. For the remainder of this section, we introduce the following notation: let $\Gamma_K := \operatorname{Gal}(K^{\text{sep}}/K)$ denote the absolute Galois group of K. For every maximal ideal $\mathfrak{p}$ of A, denote the $\mathfrak{p}$-adic completions of A and F by $A_{\mathfrak{p}}$ and $F_{\mathfrak{p}}$.

Definition 2.25.

(a) An *integral* $\mathfrak{p}$*-adic Galois representation* is a free $A_{\mathfrak{p}}$-module of finite rank together with a continuous group homomorphism $\rho\colon \Gamma_K \to \operatorname{Aut}_{A_{\mathfrak{p}}}(V)$. Equipped with Γ_K-equivariant $A_{\mathfrak{p}}$-linear homomorphisms, we obtain the category $\operatorname{Rep}_{A_{\mathfrak{p}}}(\Gamma_K)$ of integral $\mathfrak{p}$-adic Galois representations.

(b) A *rational* $\mathfrak{p}$*-adic Galois representation* is a finite-dimensional $F_{\mathfrak{p}}$-vector space together with a continuous group homomorphism $\rho\colon \Gamma_K \to \operatorname{Aut}_{F_{\mathfrak{p}}}(V)$. Equipped with Γ_K-equivariant $A_{\mathfrak{p}}$-linear homomorphisms, we obtain the category $\operatorname{Rep}_{F_{\mathfrak{p}}}(\Gamma_K)$ of rational $\mathfrak{p}$-adic Galois representations.

Definition 2.26. Let $\mathfrak{p} \neq \ker\iota$ be a maximal ideal of A, and let $A_{K^{\text{sep}},\mathfrak{p}} := \varprojlim_n((A/\mathfrak{p}^n) \otimes_{\mathbb{F}_q} K^{\text{sep}})$ denote the completion of $A \otimes_{\mathbb{F}_q} K^{\text{sep}}$ at $\mathfrak{p}$. For every A-motive $X = (M, L)$ over K:

(a) The *integral Tate module* of X at $\mathfrak{p}$ is the $A_{\mathfrak{p}}$-module

$$\mathrm{T}_{\mathfrak{p}}(X) := (A_{K^{\text{sep}},\mathfrak{p}} \otimes_{A_K} M)^{\tau} \otimes_{A_{\mathfrak{p}}} ((A_{K^{\text{sep}},\mathfrak{p}} \otimes_{A_K} L)^{\tau})^{\vee},$$

with τ-invariants taken with respect to the natural diagonal σ-linear endomorphism, equipped with the induced action of Γ_K.

(b) The *rational Tate module* of X at $\mathfrak{p}$ is the $F_{\mathfrak{p}}$-vector space

$$\mathrm{V}_{\mathfrak{p}}(X) := F_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} \mathrm{T}_{\mathfrak{p}}(X),$$

equipped with the induced action of Γ_K.

Definition 2.27. Let $R \to S$ be a homomorphism of unital rings, $\mathcal{C}$ an R-linear category, and $\mathcal{D}$ an S-linear category. An R-linear functor $V : \mathcal{C} \to \mathcal{D}$ is *S/R-faithful* (respectively, *S/R-fully faithful*) if the natural homomorphism

$$S \otimes_R \operatorname{Hom}_{\mathcal{C}}(X, Y) \to \operatorname{Hom}_{\mathcal{D}}(VX, VY)$$

is injective (respectively, bijective) for all objects X, Y of $\mathcal{C}$.

Proposition 2.28. *Let $\mathfrak{p} \neq \ker\iota$ be a maximal ideal of A.*

(a) *The functor $\mathrm{T}_{\mathfrak{p}}$ is an A-linear tensor functor with values in integral $\mathfrak{p}$-adic representations, which is $A_{\mathfrak{p}}/A$-faithful and preserves ranks.*

(b) *The functor $\mathrm{V}_{\mathfrak{p}}$ extends uniquely to an F-linear functor with values in rational $\mathfrak{p}$-adic representations, again denoted as $\mathrm{V}_{\mathfrak{p}}$, such that the following diagram commutes.*

$$\begin{tikzcd}
A\text{-Mot}_K \arrow[r, "\mathrm{T}_{\mathfrak{p}}"] \arrow[d] & \mathrm{Rep}_{A_{\mathfrak{p}}}(\Gamma_K) \arrow[d, "F_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} (-)"] \\
A\text{-Isomot}_K \arrow[r, "\mathrm{V}_{\mathfrak{p}}"] & \mathrm{Rep}_{F_{\mathfrak{p}}}(\Gamma_K)
\end{tikzcd}$$

(c) *The functor $\mathrm{V}_{\mathfrak{p}}$ is an exact F-linear tensor functor which is $F_{\mathfrak{p}}/F$-faithful and preserves ranks.*

Proof.

(a) Let us first consider the restriction of $\mathrm{T}_{\mathfrak{p}}$ to effective A-motives, it maps a given effective A-motive M to

$$\mathrm{T}_{\mathfrak{p}}(M) = (A_{K^{\mathrm{sep}},\mathfrak{p}} \otimes_{A_K} M)^{\tau} = \varprojlim_n ((M \otimes_K K^{\mathrm{sep}})/\mathfrak{p}^n)^{\tau}.$$

Note that the assumption that $\mathfrak{p} \neq \ker \iota$ implies that the linearisation of the σ-linear endomorphism of $(M \otimes_K K^{\mathrm{sep}})/\mathfrak{p}^n$ is bijective. By applying Proposition 2.16 to K^{sep} and $(M \otimes_K K^{\mathrm{sep}})/\mathfrak{p}^n$, we see that $((M \otimes_K K^{\mathrm{sep}})/\mathfrak{p}^n)^{\tau}$ is a free $A/\mathfrak{p}^n$-module of rank $\mathrm{rk}(M)$. It follows that $\mathrm{T}_{\mathfrak{p}}(M)$ is an integral $\mathfrak{p}$-adic Galois representation of rank $\mathrm{rk}(M)$. Using Proposition 2.16 again, it follows that the restriction of $\mathrm{T}_{\mathfrak{p}}$ to $A\text{-Mot}_K^{\mathrm{eff}}$ is an A-linear tensor functor with values in integral $\mathfrak{p}$-adic representations which preserves ranks. By construction, this implies that $\mathrm{T}_{\mathfrak{p}}$ itself has these properties.

It remains to show that $\mathrm{T}_{\mathfrak{p}}$ is $A_{\mathfrak{p}}/A$-faithful. Let M, N be A-motives. We may assume that both are effective. Note that we have a natural inclusion $A_{\mathfrak{p}} \otimes_{\mathbb{F}_q} K^{\mathrm{sep}} \subset A_{K^{\mathrm{sep}},\mathfrak{p}}$. It follows that we have a natural inclusion

$$(A_{\mathfrak{p}} \otimes K^{\mathrm{sep}}) \otimes_{A_K} \mathrm{Hom}_{A_K}(M, N) \subset A_{K^{\mathrm{sep}},\mathfrak{p}} \otimes_{A_K} \mathrm{Hom}_{A_K}(M, N).$$

On both sides, the left exact functors $(-)^{\Gamma_K}$ of Galois-invariants and $(-)^{\tau} := \ker(\tau_N \circ (-) - (-) \circ \tau_M)$ of τ-invariants act, and the two actions commute. Therefore,

$$((A_{\mathfrak{p}} \otimes K^{\mathrm{sep}}) \otimes_{A_K} \mathrm{Hom}_{A_K}(M, N))^{\Gamma_K,\tau} \subset (A_{K^{\mathrm{sep}},\mathfrak{p}} \otimes_{A_K} \mathrm{Hom}_{A_K}(M, N))^{\tau,\Gamma_K},$$

so

$$A_{\mathfrak{p}} \otimes_A \mathrm{Hom}(M, N)^{\tau} \subset \mathrm{Hom}_A(\mathrm{T}_{\mathfrak{p}}\, M, \mathrm{T}_{\mathfrak{p}}\, N)^{\Gamma_K},$$

which means that $A_{\mathfrak{p}} \otimes_A \mathrm{Hom}(M, N) \to \mathrm{Hom}_{\Gamma_K}(\mathrm{T}_{\mathfrak{p}}\, M, \mathrm{T}_{\mathfrak{p}}\, N)$ is injective, as desired.

(b) Since scalar isogenies are mapped to isomorphisms in $\mathrm{Rep}_{F_{\mathfrak{p}}}(\Gamma_K)$, $\mathrm{V}_{\mathfrak{p}}$ extends to an F-linear functor on $A\text{-Isomot}_K$ with values in rational $\mathfrak{p}$-adic Galois representations.

(c) Now item (a) implies that $\mathrm{V}_{\mathfrak{p}}$ is an $F_{\mathfrak{p}}/F$-fully faithful tensor functor, and preserves ranks. This last property implies that $\mathrm{V}_{\mathfrak{p}}$ is exact. □

COROLLARY 2.29.

(a) *For every two A-motives M, N, the A-module of homomorphisms $\mathrm{Hom}_{A\text{-Mot}_K}(M, N)$ is finitely generated and projective.*

(b) *For every two A-isomotives X, Y, the F-vector space of homomorphisms $\mathrm{Hom}_{A\text{-Isomot}_K}(X, Y)$ is finite-dimensional.*

(c) *Every A-isomotive has a composition series of finite length.*

Proof.

(b) Since $\mathrm{Hom}_{\Gamma_K}(\mathrm{V}_{\mathfrak{p}}\, X, \mathrm{V}_{\mathfrak{p}}\, Y)$ is finite $F_{\mathfrak{p}}$-dimensional, so is $F_{\mathfrak{p}} \otimes_F \mathrm{Hom}_{A\text{-Isomot}_K}(X, Y)$ by $F_{\mathfrak{p}}/F$-faithfulness of $\mathrm{V}_{\mathfrak{p}}$. This implies the desired statement.

(a) If we show that $\mathrm{Hom}_{A\text{-Mot}_K}(M, N)$ is torsion-free, item (a) follows from item (b). However, $\mathrm{Hom}_{A\text{-Mot}_K}(M, N) = (M^\vee \otimes N)^\tau$ is a submodule of the torsion-free A-module $M^\vee \otimes N$, so we are done.

(c) Since $\mathrm{V}_{\mathfrak{p}}$ is faithful, it maps non-zero objects to non-zero objects. Therefore, the length of an A-isomotive is bounded by the length of its Tate module. Since the latter is of finite length, so is the former. □

3. Some semilinear algebra

We begin this section by introducing the notion 'semisimple on objects' for functors, a categorical generalisation of the statement of Theorem 1.1, and discuss how this property combines with the notion of 'relative full faithfulness', introduced in Definition 2.27.

We then introduce some terminology for semilinear algebra, and prove a theorem on bold scalar extension of restricted modules for a certain class of bold rings. The reader may choose to skip to § 4 after reading the statement of Theorem 3.11, to see how it is employed.

Definition 3.1. Let $\mathcal{A}, \mathcal{B}$ be abelian categories. An exact functor $V : \mathcal{A} \to \mathcal{B}$ is *semisimple on objects* if it maps semisimple objects of $\mathcal{A}$ to semisimple objects of $\mathcal{B}$.

We intersperse a proposition which exemplifies nicely how the properties of being 'relatively' full faithful and being semisimple on objects combine.

Proposition 3.2. *Let F'/F be a field extension, $\mathcal{A}$ an F-linear abelian category, and $\mathcal{B}$ an F'-linear abelian category. Consider an F'/F-fully faithful F-linear exact functor $V : \mathcal{A} \to \mathcal{B}$. For every object X of $\mathcal{A}$, if $V(X)$ is semisimple in $\mathcal{B}$, then X is semisimple*[6] *in $\mathcal{A}$.*

Proof. Assume that

$$\alpha : \quad 0 \to X' \to X \to X'' \to 0$$

is a short exact sequence in $\mathcal{A}$ such that the exact sequence $V(\alpha)$ splits in $\mathcal{B}$. We must show that α splits, and for this it suffices to show that $\mathrm{id}_{X''}$ is in the image of the natural homomorphism $\mathrm{Hom}_{\mathcal{A}}(X'', X) \to \mathrm{Hom}_{\mathcal{A}}(X'', X'')$. This image coincides with the intersection of $\mathrm{Hom}_{\mathcal{A}}(X'', X'')$ and the image of the natural homomorphism $F' \otimes_F \mathrm{Hom}_{\mathcal{A}}(X'', X) \to F' \otimes_F \mathrm{Hom}_{\mathcal{A}}(X'', X'')$. By F'/F-full faithfulness, we may identify this latter image with the image of the natural homomorphism $\mathrm{Hom}_{\mathcal{B}}(V(X''), V(X)) \to \mathrm{Hom}_{\mathcal{B}}(V(X''), V(X''))$. By assumption, $\mathrm{id}_{V(X'')} = V(\mathrm{id}_{X''})$ is an element of this image, and under our natural identifications it is also clearly an element of $\mathrm{Hom}_{\mathcal{A}}(X'', X'')$, therefore we are done. □

We turn to some general terminology for semilinear algebra. Other authors (Anderson, Pink, Taguchi, Tamagawa and, in particular, Fontaine) have used different names in different contexts, such as φ-, σ- and τ-modules. We choose to use terminology that abstracts from the particular context and choice of notation, so as to prove the basic properties of these objects in suitable generality.

[6] In other words, V maps non-semisimple objects of $\mathcal{A}$ to non-semisimple objects of $\mathcal{B}$; we use this in the proof of Proposition 3.18.

DEFINITION 3.3. A *bold ring* $\boldsymbol{R}$ is a unital commutative ring R equipped with a unital ring endomorphism $\sigma\colon R \to R$. The *coefficient ring* of $\boldsymbol{R}$ is its subring $R^\sigma := \{r \in R\colon \sigma(r) = r\}$ of σ-invariant elements.

A *homomorphism* $\boldsymbol{S} \to \boldsymbol{R}$ of bold rings is a ring homomorphism that commutes with σ. It induces a homomorphism $S^\sigma \to R^\sigma$ of coefficient rings.

DEFINITION 3.4. Let $\boldsymbol{R}$ be a bold ring. A (bold) *$\boldsymbol{R}$-module* $\boldsymbol{M}$ is an R-module M together with a σ-linear endomorphism $\tau\colon M \to M$.

A *homomorphism* $\boldsymbol{M} \to \boldsymbol{N}$ of $\boldsymbol{R}$-modules is an R-module homomorphism that commutes with τ. The *tensor product* $\boldsymbol{M} \otimes_{\boldsymbol{R}} \boldsymbol{N}$ of $\boldsymbol{M} = (M, \tau_M)$ and $\boldsymbol{N} = (N, \tau_N)$ is the R-module $M \otimes_R N$ together with the σ-linear endomorphism

$$M \otimes_R N \to M \otimes_R N, \quad m \otimes n \mapsto \tau_M(m) \otimes \tau_N(n).$$

The category $\boldsymbol{R}$-Mod of $\boldsymbol{R}$-modules is an R^σ-linear abelian tensor category.

DEFINITION 3.5. Let $\boldsymbol{S} \xrightarrow{f} \boldsymbol{R}$ be a homomorphism of bold rings. *Bold scalar extension* from $\boldsymbol{S}$ to $\boldsymbol{R}$ is the functor $\boldsymbol{S}$-Mod $\to$ $\boldsymbol{R}$-Mod mapping an $\boldsymbol{S}$-module $\boldsymbol{M}$ to $\boldsymbol{R} \otimes_{\boldsymbol{S}} \boldsymbol{N}$ and a homomorphism h of $\boldsymbol{S}$-modules to $\mathrm{id}_R \otimes h$.

Recall from § 2 that the σ-linear endomorphism τ of a module $\boldsymbol{M}$ over a bold ring $\boldsymbol{R} = (R, \sigma)$ corresponds to a unique R-linear homomorphism $\tau_{\mathrm{lin}}\colon \sigma_* M := R \otimes_{\sigma,R} M \to M$, its linearisation.

DEFINITION 3.6. Let $\boldsymbol{R}$ be a bold ring.

(a) An $\boldsymbol{R}$-module $\boldsymbol{M} = (M, \tau)$ is *restricted* if M is a finitely generated projective R-module and τ_{lin} is bijective.

(b) Let $\boldsymbol{S} \xrightarrow{f} \boldsymbol{R}$ be a homomorphism of bold rings. An $\boldsymbol{R}$-module $\boldsymbol{M}$ is *f-restricted* if there exist a restricted $\boldsymbol{S}$-module $\boldsymbol{N}$ and an isomorphism $\boldsymbol{M} \cong \boldsymbol{R} \otimes_{\boldsymbol{S}} \boldsymbol{N}$ of $\boldsymbol{R}$-modules. Clearly, this implies that $\boldsymbol{M}$ is restricted in the sense of statement (a).

Other authors use the terminology *étale* for what we call restricted. This author finds that analogy a little far-fetched, and not specific enough if one has to deal with several rings, as we do here.

Let $\mathbb{F}_q, K, \sigma_q$ be as in § 2, so $\mathbb{F}_q$ is a finite field and K is a field containing $\mathbb{F}_q$. In this section (but not the next) $F/\mathbb{F}_q$ may be any field extension, that is, $\boxed{\text{we } \textit{drop} \text{ the assumption that } F \text{ is a global field}}$. In addition to yielding more generality, this allows us more flexibility in the proofs.

Let $F_K = \mathrm{Frac}(F \otimes_{\mathbb{F}_q} K)$ denote the total ring of fractions of $F \otimes_{\mathbb{F}_q} K$. The bold ring $\boldsymbol{F_K}$ is given by F_K together with the endomorphism $\sigma = \sigma_{F_K} = \mathrm{Frac}(\mathrm{id} \otimes \sigma_q)$ induced by σ_q. If F'/F is a field extension, the bold ring $\boldsymbol{F'_K}$ with underlying ring $F'_K = \mathrm{Frac}(F' \otimes_{\mathbb{F}_q} K)$ is defined analogously, and we have a bold scalar extension functor $\boldsymbol{F'_K} \otimes_{\boldsymbol{F_K}} (-)$ from $\boldsymbol{F_K}$-modules to $\boldsymbol{F'_K}$-modules.

LEMMA 3.7. *Assume that the number of roots of unity of K is finite.*

(a) *The ring F_K is a finite product of pairwise isomorphic fields.*

(b) *The underlying F_K-module of every restricted $\boldsymbol{F_K}$-module is free.*

Proof. Let $\mathbb{F}_F$ and $\mathbb{F}_K$ denote the algebraic closures of $\mathbb{F}_q$ in F and K, respectively. If $\mathbb{F}_F = \mathbb{F}_{q^r}$ and $\mathbb{F}_K = \mathbb{F}_{q^s}$ are both finite, then

$$\mathbb{F}_F \otimes_{\mathbb{F}_q} \mathbb{F}_K \cong (\mathbb{F}_{q^{\mathrm{lcm}(r,s)}})^{\times \gcd(r,s)}$$

and $\sigma = \mathrm{id} \otimes \sigma_q$ corresponds to an endomorphism of the product which permutes the factors transitively. This implies that every restricted $(\mathbb{F}_F \otimes_{\mathbb{F}_q} \mathbb{F}_K, \mathrm{id} \otimes \sigma_q)$-module has an underlying $\mathbb{F}_F \otimes_{\mathbb{F}_q} \mathbb{F}_K$-module which is projective of *constant* rank, and hence free. Hereby, items (a) and (b) are proven for F and K both finite.

If $\mathbb{F}_F$ is infinite, then it is an algebraic closure of $\mathbb{F}_q$ and

$$\mathbb{F}_F \otimes_{\mathbb{F}_q} \mathbb{F}_K \cong (\mathbb{F}_F)^{\times \dim_{\mathbb{F}_q} \mathbb{F}_K}$$

is a product of pairwise isomorphic fields. It follows from the above that the endomorphism corresponding to $\sigma = \mathrm{id} \otimes \sigma_q$ permutes the factors transitively, so again we have items (a) and (b) for F and K both algebraic.

In the general case, $\mathbb{F}_F \otimes_{\mathbb{F}_q} \mathbb{F}_K \cong \mathbb{F}^r$ for an algebraic extension $\mathbb{F}/\mathbb{F}_q$ and an integer $r \geqslant 1$. Then [Jac90, Theorem 8.50] shows that $F \otimes_{\mathbb{F}_F} \mathbb{F} \otimes_{\mathbb{F}_K} K$ is a domain, which implies that

$$F_K \cong \mathrm{Frac}(F \otimes_{\mathbb{F}_F} \mathbb{F} \otimes_{\mathbb{F}_K} K)^{\times r}$$

is a product of pairwise isomorphic fields. Tracing through these identifications, we see that σ_{F_K} permutes these fields transitively, so we obtain items (a) and (b) in general. □

Proposition 3.8. *Assume that the number of roots of unity of K is finite. The full subcategory of restricted $\boldsymbol{F_K}$-modules is closed under subquotients and tensor products in the category of all $\boldsymbol{F_K}$-modules. In particular, it is an F-linear rigid abelian tensor category.*

Proof. Let $\boldsymbol{M} = (M, \tau)$ be a restricted $\boldsymbol{F_K}$-module, and consider an exact sequence

$$0 \to (M', \tau') \to \boldsymbol{M} \to (M'', \tau') \to 0$$

of $\boldsymbol{F_K}$-modules. Both M' and M'' are finitely generated F_K-modules since F_K is Noetherian, and both are projective F_K-modules since F_K is a product of fields by Lemma 3.7(a). Since $\tau_{\mathrm{lin}} : \sigma_* M \to M$ is bijective, the Snake lemma implies that τ'_{lin} is injective and τ''_{lin} is surjective. By Lemma 3.7, this implies that both τ'_{lin} and τ''_{lin} are bijective. Therefore, both (M', τ') and (M'', τ'') are restricted $\boldsymbol{F_K}$-modules as claimed.

We suppress the easy proof that the tensor product of restricted $\boldsymbol{F_K}$-modules is restricted. It follows that the full subcategory of restricted $\boldsymbol{F_K}$-modules is an F-linear abelian tensor category, since $\boldsymbol{F_K}$-Mod is. One checks that the dual of a restricted $\boldsymbol{F_K}$-module (M, τ) is given by $M^\vee := \mathrm{Hom}_{A_K}(M, A_K)$ together with the σ-linear endomorphism mapping $f \in M^\vee$ to $\tau_{M^\vee}(f) := \sigma_{\mathrm{lin}} \circ \sigma_*(f) \circ (\tau_{\mathrm{lin}})^{-1}$. It follows that the category of restricted $\boldsymbol{F_K}$-modules is a rigid tensor category. □

We turn to the main theorem of this section, its proof will occupy the remainder of the section. To state it, we recall the algebraic concept of separability.

Definition 3.9. A field extension F'/F is *separable* if for every field extension $F'' \supset F$ the ring $F' \otimes_F F''$ is reduced (contains no nilpotent elements).

Remark 3.10. An algebraic field extension F'/F is separable in the sense of Definition 3.9 if and only if it is separable in the usual sense. If $F''/F'/F$ is a tower of field extensions such that F''/F is separable, then F'/F is separable as well.

THEOREM 3.11. *Let $F'/F/\mathbb{F}_q$ be a tower of field extensions. Assume that the number of roots of unity of K is finite. The restriction of the functor of bold scalar extension $\boldsymbol{F'_K} \otimes_{\boldsymbol{F_K}} (-)$ to restricted $\boldsymbol{F_K}$-modules is:*

(a) *F'/F-fully faithful; and*

(b) *if F'/F is a separable field extension, it is semisimple on objects.*

We turn first to the proof of Theorem 3.11(a).

PROPOSITION 3.12. *Let $F'/F/\mathbb{F}_q$ be a tower of field extensions. Assume that the number of roots of unity of K is finite. The restriction of the functor of bold scalar extension $\boldsymbol{F'_K} \otimes_{\boldsymbol{F_K}} (-)$ to restricted $\boldsymbol{F_K}$-modules is F'/F-fully faithful.*

Proof. Let $\boldsymbol{M}, \boldsymbol{N}$ be restricted $\boldsymbol{F_K}$-modules, and set $\boldsymbol{X} := \boldsymbol{M}^\vee \otimes_{\boldsymbol{F_K}} \boldsymbol{N}$. Since $\mathrm{Hom}_{\boldsymbol{F_K}}(\boldsymbol{M}, \boldsymbol{N}) = \boldsymbol{X}^\tau$ and $\mathrm{Hom}_{\boldsymbol{F'_K}}(\boldsymbol{F'_K} \otimes_{\boldsymbol{F_K}} \boldsymbol{M}, \boldsymbol{F'_K} \otimes_{\boldsymbol{F_K}} \boldsymbol{N}) = (\boldsymbol{F'_K} \otimes_{\boldsymbol{F_K}} \boldsymbol{X})^\tau$, it is sufficient to prove that

$$F' \otimes_F \boldsymbol{X}^\tau \to (\boldsymbol{F'_K} \otimes_{\boldsymbol{F_K}} \boldsymbol{X})^\tau \tag{3.13}$$

is bijective for all restricted $\boldsymbol{F_K}$-modules $\boldsymbol{X}$. We set $\boldsymbol{X'} := \boldsymbol{F'_K} \otimes_{\boldsymbol{F_K}} \boldsymbol{X}$.

Since the homomorphism $F' \otimes_F F_K \to F'_K = \mathrm{Frac}(F' \otimes_F F_K)$ is injective and the functor $(-)^\tau$ is left-exact, the homomorphism of (3.13) is injective. We must show that it is surjective.

Moreover, we may assume that $F' \supset F$ is finitely generated, since for every element $x' \in (\boldsymbol{X'})^\tau$ there exists a finitely generated field extension $F' \supset F^0 \supset F$ such that x' lies in $(\boldsymbol{X^0})^\tau$, where $\boldsymbol{X^0} := \boldsymbol{F^0_K} \otimes_{\boldsymbol{F_K}} \boldsymbol{X}$ with $\boldsymbol{F^0_K} := \mathrm{Frac}(F^0 \otimes_F F_K, \mathrm{id} \otimes \sigma_q)$.

All in all, the theorem reduces to proving the surjectivity of (3.13) for the two special cases of $F' \supset F$ finite, and $F' \supset F$ purely transcendental of transcendence degree one. The first is easy, since if F'/F is finite, then $F' \otimes_F \boldsymbol{F_K} \cong \boldsymbol{F'_K}$, and hence

$$F' \otimes_F \boldsymbol{M}^\tau = (F' \otimes_F \boldsymbol{F_K} \otimes_{\boldsymbol{F_K}} \boldsymbol{M})^\tau \cong (\boldsymbol{F'_K} \otimes_{\boldsymbol{F_K}} \boldsymbol{M})^\tau$$

as claimed. The second is dealt with in the following Proposition 3.14. □

PROPOSITION 3.14. *If $F' = F(X)$ is a purely transcendental extension of F of transcendence degree one and $\boldsymbol{X}$ is a restricted $\boldsymbol{F_K}$-module, then $F' \otimes_F \boldsymbol{X}^\tau \to (\boldsymbol{F'_K} \otimes_{\boldsymbol{F_K}} \boldsymbol{X})^\tau$ is surjective.*

For the proof of Proposition 3.14, we use a slightly extended notion of 'denominators'. By Lemma 3.7(a), the ring $F_K = Q^{\times s}$ for some field Q. We set $F_K[X] := Q[X]^{\times s}$ and $F_K(X) := \mathrm{Frac}(F(X) \otimes_F F_K) = Q(X)^{\times s}$.

For $f \in F_K(X)$, we define the *denominator* $\mathrm{den}(f)$ of f componentwise, as the s-tuple of the usual (monic) denominators of its s components. Similarly, for $f, g \in F_K(X)$, we define the *least common multiple* $\mathrm{lcm}(f, g)$ of f and g componentwise, as the s-tuple of the usual (monic) least common multiples of their corresponding components.

Clearly, for $f, g \in F_K(X)$ the following relation holds, where $\mid$ denotes componentwise divisibility in $F_K[X]$:

$$\mathrm{den}(f+g) \mid \mathrm{lcm}(\mathrm{den}\, f, \mathrm{den}\, g). \tag{3.15}$$

We may now characterise the subring $F(X) \otimes_F F_K$ of $F_K(X)$.

LEMMA 3.16. *We have*

$$F(X) \otimes_F F_K = \left\{ f \in F_K(X) : \begin{array}{c} \mathrm{den}(f) \mid g \\ \textit{for some } g \in F[X] \smallsetminus \{0\} \end{array} \right\}.$$

Proof. $\subset$: Assume that f is an element of $F(X)\otimes_F F_K$. We may write $f=\sum_{i=1}^m(a_i/b_i)\otimes\lambda_i$ for elements $\lambda_i\in F_K$ and $a_i,b_i\in F[X]$ with $b_i\neq 0$. By (3.15), $\mathrm{den}(f)$ divides $d:=\prod_{i=1}^m b_i$, an element of $F[X]\smallsetminus\{0\}$ as claimed.

$\supset$: Assume that f is an element of $F_K(X)$ which divides a non-zero element $g\in F[X]$. This means that there exists an element $h\in F_K[X]$ such that $g=\mathrm{den}(f)\cdot h$. We have $f=f'/\mathrm{den}(f)$ for $f':=f\,\mathrm{den}(f)\in F_K[X]$. Therefore, $f=(f'h)/(\mathrm{den}(f)h)$ with $1/(\mathrm{den}(f)h)=1/g\in F(X)$ and $f'h\in F_K[X]\subset F(X)\otimes_F F_K$, which implies our claim that f is an element of $F(X)\otimes_F F_K$. $\square$

Given a vector $v=(v_j)\in F_K(X)^r$ for some $r\geqslant 1$, we set $\mathrm{den}(v)=\mathrm{lcm}_j(\mathrm{den}\,v_j)$.

Lemma 3.17. *Given two integers $m,n\geqslant 1$, a matrix $A\in\mathrm{Mat}_{m\times n}(F_K)$ and a vector $v\in F_K(X)^{\oplus n}$, we have*

$$\mathrm{den}(Av)\mid\mathrm{den}(v).$$

In particular, if $m=n$ and A is invertible, then $\mathrm{den}(Av)=\mathrm{den}(v)$.

Proof. We suppress the easy proof of the divisibility statement, which is clear intuitively.

In the case $m=n$ and A is invertible, we may additionally apply this divisibility statement to the matrix A^{-1} and the vector Av. We obtain $\mathrm{den}(v)=\mathrm{den}(A^{-1}(Av))\mid\mathrm{den}(Av)$. Since both $\mathrm{den}(Av)$ and $\mathrm{den}(v)$ have monic components, we infer that $\mathrm{den}(Av)=\mathrm{den}(v)$. $\square$

Proof of Proposition 3.14. By Lemma 3.7(a), $X=F_K^r$ for an integer $r\geqslant 0$ and $\tau=\Delta\circ\sigma$ for a certain matrix $\Delta\in\mathrm{GL}_r(F_K)$.

Assume that $x'\in F_K(X)\otimes_{F_K}X$ is τ-invariant, so $x'=(x'_i)_i\in F_K(X)^r$ and $x'=\Delta(\sigma(x'))$. By Lemma 3.17 applied to the invertible matrix Δ and the vector $\sigma(x')$, we obtain that $\mathrm{den}(x')=\mathrm{den}(\sigma(x'))$, and this latter vector clearly coincides with $\sigma(\mathrm{den}(x'))$. Therefore, $\mathrm{den}(x')=\sigma(\mathrm{den}(x'))$ is an element of $F[X]$. Since $\mathrm{den}(x'_i)\mid\mathrm{den}(x')$ by definition, all x'_i are elements of $F'\otimes_F F_K$ by Lemma 3.16, and so $x'\in F'\otimes_F \boldsymbol{X}^\tau$, as claimed. $\square$

We now turn to the proof of Theorem 3.11(b).

Proposition 3.18. *Let $F'/F/\mathbb{F}_q$ be a tower of field extensions. Assume that F'/F is separable and the number of roots of unity of K is finite. The restriction of the functor of bold scalar extension $\boldsymbol{F'_K}\otimes_{\boldsymbol{F_K}}(-)$ to restricted $\boldsymbol{F_K}$-modules is semisimple on objects.*

Proof. As in the proof of Proposition 3.12, we start by reducing to the case where F'/F is finitely generated: if $\boldsymbol{M}$ is a semisimple restricted $\boldsymbol{F_K}$-module but $\boldsymbol{M'}:=\boldsymbol{F'_K}\otimes_{\boldsymbol{F_K}}\boldsymbol{M}$ is not semisimple, then there exists a non-split short exact sequence

$$0\to\boldsymbol{M}'_1\xrightarrow{f}\boldsymbol{M}'\xrightarrow{g}\boldsymbol{M}'_2\to 0. \tag{3.19}$$

Clearly, there exists a finitely generated field extension, $F'\supset F^0\supset F$ such that $\boldsymbol{M}'_1,\boldsymbol{M}'_2,f,g$ are defined over $\boldsymbol{F^0_K}=\mathrm{Frac}(F^0\otimes_F F_K,\mathrm{id}\otimes\sigma_q)$. The short exact sequence inducing (3.19) must be non-split by Propositions 3.2 and 3.12. Thereby, we would find a contradiction to Proposition 3.18 for finitely generated field extensions. Note that F^0/F is separable since F'/F is.

The same argument shows that the proof of our proposition reduces to the special cases of finite separable field extensions and purely transcendental field extensions of transcendence degree one. We deal with these cases separately in the following two propositions. Note that it is sufficient to show that the bold scalar extension of a simple restricted $\boldsymbol{F_K}$-module is semisimple, since bold scalar extension is an additive functor. $\square$

PROPOSITION 3.20. *Assume that the number of roots of unity of K is finite. Let $F'/F/\mathbb{F}_q$ be a tower of field extensions such that F'/F is finite separable, and let $\boldsymbol{M}$ be a simple restricted $\boldsymbol{F_K}$-module. Then $\boldsymbol{M}' := \boldsymbol{F'_K} \otimes_{\boldsymbol{F_K}} \boldsymbol{M}$ is semisimple.*

Proof. We start with the case where F'/F is a finite *Galois* extension, and set $\Gamma := \mathrm{Gal}(F'/F)$. Assume that $\boldsymbol{S}' \subset \boldsymbol{M}'$ is a simple $\boldsymbol{F'_K}$-submodule. Set

$$\boldsymbol{X}' := \sum_{g\in\Gamma} g\boldsymbol{S}' \subset \boldsymbol{M}'.$$

This $\boldsymbol{F'_K}$-module is Γ-invariant, so $\boldsymbol{X}' = \boldsymbol{F'_K} \otimes_{\boldsymbol{F_K}} \boldsymbol{X}$ for some $\boldsymbol{F_K}$-submodule of $\boldsymbol{X} \subset \boldsymbol{M}$. Since $\boldsymbol{M}$ is simple and $\boldsymbol{S}' \neq 0$, we see that $\boldsymbol{X} = \boldsymbol{M}$ and so $\boldsymbol{M}' = \sum_{g\in\Gamma} g\boldsymbol{S}'$ is semisimple as a sum of simple objects.

In the general case, let F''/F denote a Galois closure of F'/F, and consider a simple restricted $\boldsymbol{F_K}$-module $\boldsymbol{M}$. By what we have proven, $\boldsymbol{M}'' := \boldsymbol{F''_K} \otimes_{\boldsymbol{F_K}} \boldsymbol{M}$ is semisimple. Now Proposition 3.2 shows that $\boldsymbol{M}' := \boldsymbol{F'_K} \otimes_{\boldsymbol{F_K}} \boldsymbol{M}$ is semisimple, since we have already proven Proposition 3.12. □

PROPOSITION 3.21. *Assume that the number of roots of unity of K is finite. Let $F/\mathbb{F}_q$ be a field extension, consider $F' = F(X)$ and let $\boldsymbol{M}$ be a simple restricted $\boldsymbol{F_K}$-module. Then $\boldsymbol{M}' := \boldsymbol{F'_K} \otimes_{\boldsymbol{F_K}} \boldsymbol{M}$ is simple.*

Proof. Recall that $F_K = Q^s$ for some field Q by Lemma 3.7(a), so $F_K(X) := F'_K = Q(X)^s$. Let $\boldsymbol{F_K}[X]$ be the bold ring consisting of $F_K[X] = Q[X]^s$ together with the restriction of $\sigma_{F'_K}$; it acts as the identity on X. Now $\boldsymbol{\mathcal{M}} := \boldsymbol{F_K}[X] \otimes_{\boldsymbol{F_K}} \boldsymbol{M}$ is a 'model' of $\boldsymbol{M}'$ in the sense that $\boldsymbol{\mathcal{M}}$ is a restricted $\boldsymbol{F_K}[X]$-module such that $\boldsymbol{M}' = \boldsymbol{F_K}(X) \otimes_{\boldsymbol{F_K}[X]} \boldsymbol{\mathcal{M}}$. Moreover, $\boldsymbol{M} = \boldsymbol{\mathcal{M}}/X$.

Assume that $\boldsymbol{M}'$ is not simple, so there exists a non-trivial $\boldsymbol{F'_K}$-submodule $\boldsymbol{N}' \subsetneq \boldsymbol{M}'$. It follows that $\boldsymbol{\mathcal{N}} := \boldsymbol{\mathcal{M}} \cap \boldsymbol{N}'$ is a non-trivial $\boldsymbol{F_K}[X]$-submodule of $\boldsymbol{\mathcal{M}}$ other than $\boldsymbol{\mathcal{M}}$, and therefore that $\boldsymbol{N} := \boldsymbol{\mathcal{N}}/(X)$ is a non-trivial $\boldsymbol{F_K}$-submodule of $\boldsymbol{\mathcal{M}}/(X) \cong \boldsymbol{M}$ other than $\boldsymbol{M}$. This contradicts the simplicity of $\boldsymbol{M}$, using Proposition 3.8. □

Proof of Theorem 3.11. Proposition 3.12 gives item (a), and Proposition 3.18 gives item (b). □

4. Translation to semilinear algebra

In this section, we embed the categories of A-motives and A-isomotives in categories of bold modules, and classify the categories of integral and rational $\mathfrak{p}$-adic Galois representations in terms of categories of bold modules.

This allows us to factor the functors induced by the integral and rational Tate module functors as composites of two bold scalar extension functors each. The section ends with a proof that the first factor is 'relatively' fully faithful in both cases, and semisimple on objects in the rational case.

Let $F, \mathbb{F}_q, A, K, \iota, \sigma_q$ be as in § 2. Let F_K denote the total ring of quotients $\mathrm{Frac}(F \otimes_{\mathbb{F}_q} K)$, it is a field. The bold ring $\boldsymbol{F_K}$ is given by F_K together with $\sigma = \sigma_{F_K} = \mathrm{Frac}(\mathrm{id}_F \otimes \sigma_q)$. We refer to Lemma 3.7 and Proposition 3.8 for the structure of $\boldsymbol{F_K}$ and its consequences. The bold ring $\boldsymbol{A_K} \subset \boldsymbol{F_K}$ is given by $A_K := A \otimes_{\mathbb{F}_q} K$, a Dedekind domain, together with the restriction $\sigma = \sigma_{A_K} = \mathrm{id}_A \otimes \sigma_q$ of σ_{F_K}. Given a maximal ideal $\mathfrak{p}$ of A, let $A_{(\mathfrak{p}),K}$ denote the subring of F_K consisting of those elements integral at all places $\mathfrak{P}$ of F_K lying above $\mathfrak{p}$, it is a semilocal Dedekind domain. The bold ring $\boldsymbol{A_K} \subset \boldsymbol{A_{(\mathfrak{p}),K}} \subset \boldsymbol{F_K}$ is given by $A_{(\mathfrak{p}),K}$ together with the restriction

$\sigma = \sigma_{A_{(\mathfrak{p}),K}}$ of σ_{F_K}. We say that an $\boldsymbol{F_K}$-module $\boldsymbol{M}$ is $\mathfrak{p}$-*restricted* if it is restricted with respect to the inclusion $\boldsymbol{A_{(\mathfrak{p}),K}} \subset \boldsymbol{F_K}$.

Construction 4.1. An effective A-motive (M, τ) over K induces an $\boldsymbol{F_K}$-module $\boldsymbol{F_K} \otimes_{\boldsymbol{A_K}} (M, \tau)$, which is $\mathfrak{p}$-restricted for $\mathfrak{p} \neq \ker \iota$ by the assumption that (M, τ) is of characteristic ι, and hence restricted. Thus, the essential image of the tensor functor $A\text{-Mot}_K^{\mathrm{eff}} \to \boldsymbol{A_K}\text{-Mod} \to \boldsymbol{F_K}\text{-Mod}$ consists of dualisable objects by Proposition 3.8, and so it extends uniquely to an A-linear tensor functor $I_0 \colon A\text{-Mot}_K \to \boldsymbol{F_K}\text{-Mod}$. It maps an A-motive (M, L) to $(\boldsymbol{F_K} \otimes_{\boldsymbol{A_K}} M) \otimes (\boldsymbol{F_K} \otimes_{\boldsymbol{A_K}} L)^\vee$. Now Theorem 2.23(a) implies that I_0 factors through the category of A-isomotives, so there exists a unique F-linear exact tensor functor $I \colon A\text{-Isomot}_K \longrightarrow \boldsymbol{F_K}\text{-Mod}$ such that the following diagram commutes.

$$\begin{tikzcd}
A\text{-Mot}_K^{\mathrm{eff}} \arrow[r] \arrow[d, phantom, "\cap"] & \boldsymbol{A_K}\text{-Mod} \arrow[d, "\boldsymbol{F_K} \otimes_{\boldsymbol{A_K}} (-)"] \\
A\text{-Mot}_K \arrow[r, "I_0"] \arrow[d] & \boldsymbol{F_K}\text{-Mod} \\
A\text{-Isomot}_K \arrow[ur, "I"'] &
\end{tikzcd}$$

Proposition 4.2. *The functor I is fully faithful and semisimple on objects. For every maximal ideal $\mathfrak{p} \neq \iota$ of A, the essential image of I consists of $\mathfrak{p}$-restricted $\boldsymbol{F_K}$-modules.*

Proof. The essential image of I consists of $\mathfrak{p}$-restricted $\boldsymbol{F_K}$-modules by construction.

Let us show that I is fully faithful, so let $\boldsymbol{M}, \boldsymbol{N}$ be A-isomotives. We may assume that both are effective. It is clear that

$$\mathrm{Hom}(\boldsymbol{M}, \boldsymbol{N}) \to \mathrm{Hom}_{\boldsymbol{F_K}}(\boldsymbol{F_K} \otimes_{\boldsymbol{A_K}} \boldsymbol{M}, \boldsymbol{F_K} \otimes_{\boldsymbol{A_K}} \boldsymbol{N})$$

is injective, so let h be an element of the target. Now $h(\boldsymbol{M})$ and $\boldsymbol{N}' := h(\boldsymbol{M}) \cap \boldsymbol{N}$ are effective A-motives, $h|_{\boldsymbol{M}} \colon \boldsymbol{M} \to h(\boldsymbol{M})$ is a homomorphism of effective A-motives, $h(\boldsymbol{M}) \supset \boldsymbol{N}'$ is an isogeny of effective A-motives, and $\boldsymbol{N}' \subset \boldsymbol{N}$ is a homomorphism of effective A-motives.

$$\begin{tikzcd}
\boldsymbol{F_K} \otimes_{\boldsymbol{A_K}} \boldsymbol{M} \arrow[rrr, "h"] & & & \boldsymbol{F_K} \otimes_{\boldsymbol{A_K}} \boldsymbol{N} \\
\boldsymbol{M} \arrow[u, phantom, "\cup"] \arrow[r, "h|_{\boldsymbol{M}}"] & h(\boldsymbol{M}) \supset & \boldsymbol{N}' \subset & \boldsymbol{N} \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

Now Proposition 2.20 applied to the isogeny and Theorem 2.23(a) imply that h is induced by a homomorphism $\boldsymbol{M} \to \boldsymbol{N}$ of A-isomotives.

Let us show that I is semisimple on objects, so let $\boldsymbol{M}$ be a semisimple A-isomotive. We may assume that $\boldsymbol{M}$ is effective and simple, since I is additive. Assume that $\boldsymbol{M}_0' \subset \boldsymbol{F_K} \otimes_{\boldsymbol{A_K}} \boldsymbol{M}$ is an $\boldsymbol{F_K}$-submodule. Then $\boldsymbol{M}' := \boldsymbol{M} \cap \boldsymbol{M}_0'$ is an effective A-isomotive contained in $\boldsymbol{M}$, so either $\boldsymbol{M}' = 0$ or $\boldsymbol{M}' \cong \boldsymbol{M}$ by assumption. It follows that $\boldsymbol{F_K} \otimes_{\boldsymbol{A_K}} \boldsymbol{M}$ is simple. □

We turn to two torsion-free versions of Proposition 2.16. Let K^{sep} denote a separable closure of K, with associated Galois group $\Gamma_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$. Given a maximal ideal $\mathfrak{p}$ of A, let

$$A_{K,\mathfrak{p}} := \varprojlim_n (A/\mathfrak{p}^n) \otimes_{\mathbb{F}_q} K$$

denote the completion of A_K at $\mathfrak{p}$, it is a finite product of pairwise isomorphic discrete valuation rings. Let $F_{K,\mathfrak{p}} := \mathrm{Frac}(A_{K,\mathfrak{p}})$ denote the total ring of quotients of $A_{K,\mathfrak{p}}$, it is a finite product of pairwise isomorphic fields. The bold ring $\boldsymbol{A_{K,\mathfrak{p}}}$ is given by $A_{K,\mathfrak{p}}$ together with the endomorphism $\sigma = \sigma_{A_{K,\mathfrak{p}}} = \varprojlim_n (\mathrm{id}_{A/\mathfrak{p}^n} \otimes \sigma_q)$ induced by σ_q, and the bold ring $\boldsymbol{F_{K,\mathfrak{p}}}$ is given by $F_{K,\mathfrak{p}}$ together with the endomorphism $\sigma = \sigma_{F_{K,\mathfrak{p}}} = \mathrm{Frac}(\sigma_{A_{K,\mathfrak{p}}})$ induced by σ_q. We say that an $\boldsymbol{F_{K,\mathfrak{p}}}$-module $\boldsymbol{M}$ is $\mathfrak{p}$-*restricted* if it is restricted with respect to the inclusion $\boldsymbol{A_{K,\mathfrak{p}}} \subset \boldsymbol{F_{K,\mathfrak{p}}}$.

DEFINITION 4.3.

(a) Let (M, τ) be restricted $\boldsymbol{A_{K,\mathfrak{p}}}$-module. We set

$$R'_{\mathfrak{p}}(M, \tau) := (A_{K^{\mathrm{sep}},\mathfrak{p}} \otimes_{A_{K,\mathfrak{p}}} M)^{\tau},$$

taking τ-invariants with respect to the diagonal action. Note that the action of Γ_K on $A_{K^{\mathrm{sep}},\mathfrak{p}}$ induces an action of Γ_K on $R'_{\mathfrak{p}}(T, \tau)$.

(b) Let (V, ρ) be an integral $\mathfrak{p}$-adic Galois representation. We set

$$D'_{\mathfrak{p}}(V, \rho) := (A_{K^{\mathrm{sep}},\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} V)^{\Gamma_K},$$

taking Γ_K-invariants with respect to the diagonal action. Note that the σ-linear endomorphism of $A_{K^{\mathrm{sep}},\mathfrak{p}}$ induces a σ-linear endomorphism τ of $D'_{\mathfrak{p}}(V, \rho)$.

PROPOSITION 4.4. *Let $\Gamma_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$ denote the absolute Galois group of K. The functors $D'_{\mathfrak{p}}, R'_{\mathfrak{p}}$ are quasi-inverse equivalences of $A_{\mathfrak{p}}$-linear rigid tensor categories.*

$$\left(\!\!\left(\begin{matrix}\textit{integral } \mathfrak{p}\textit{-adic}\\ \textit{Galois representations}\end{matrix}\right)\!\!\right) \underset{R'_{\mathfrak{p}}}{\overset{D'_{\mathfrak{p}}}{\rightleftarrows}} \left(\!\!\left(\begin{matrix}\textit{restricted}\\ \boldsymbol{A_{K,\mathfrak{p}}}\textit{-modules}\end{matrix}\right)\!\!\right)$$

Moreover, the following are true:

(a) $\mathrm{rk}_{A_{K,\mathfrak{p}}} D'_{\mathfrak{p}}(V, \rho) = \mathrm{rk}_{A_{\mathfrak{p}}} V$ *for every integral $\mathfrak{p}$-adic Galois representation (V, ρ);*

(b) *the homomorphism*

$$A_{K^{\mathrm{sep}},\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} R'_{\mathfrak{p}}(M, \tau) \to A_{K^{\mathrm{sep}},\mathfrak{p}} \otimes_{A_{K,\mathfrak{p}}} M$$

is an isomorphism for every restricted $\boldsymbol{A_{K,\mathfrak{p}}}$-module (M, τ);

(c) *the homomorphism*

$$A_{K^{\mathrm{sep}},\mathfrak{p}} \otimes_{A_{K,\mathfrak{p}}} D'_{\mathfrak{p}}(V, \rho) \to A_{K^{\mathrm{sep}},\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} V$$

is an isomorphism for every integral $\mathfrak{p}$-adic Galois representation (V, ρ).

Proof. This follows directly from Proposition 2.16 by considering the direct limits involved. □

DEFINITION 4.5.

(a) Let (M, τ) be $\mathfrak{p}$-restricted $\boldsymbol{F_{K,\mathfrak{p}}}$-module. We set

$$R_{\mathfrak{p}}(M, \tau) := (F_{K^{\mathrm{sep}},\mathfrak{p}} \otimes_{F_{K,\mathfrak{p}}} M)^{\tau},$$

taking τ-invariants with respect to the diagonal action. Note that the action of Γ_K on $F_{K^{\mathrm{sep}},\mathfrak{p}}$ induces an action of Γ_K on $R_{\mathfrak{p}}(T, \tau)$.

(b) Let (V, ρ) be a rational $\mathfrak{p}$-adic Galois representation. We set

$$D_{\mathfrak{p}}(V, \rho) := (F_{K^{\mathrm{sep}},\mathfrak{p}} \otimes_{F_{\mathfrak{p}}} V)^{\Gamma_K},$$

taking Γ_K-invariants with respect to the diagonal action. Note that the σ-linear endomorphism of $F_{K^{\mathrm{sep}},\mathfrak{p}}$ induces a σ-linear endomorphism τ of $D_{\mathfrak{p}}(V, \rho)$.

PROPOSITION 4.6. *Let $\Gamma_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$ denote the absolute Galois group of K. The functors $D_{\mathfrak{p}}$, $R_{\mathfrak{p}}$ are quasi-inverse equivalences of F-linear rigid abelian tensor categories.*

$$\left(\!\!\left(\begin{matrix}\textit{rational } \mathfrak{p}\textit{-adic}\\ \textit{Galois representations}\end{matrix}\right)\!\!\right) \underset{R_{\mathfrak{p}}}{\overset{D_{\mathfrak{p}}}{\rightleftarrows}} \left(\!\!\left(\begin{matrix}\mathfrak{p}\textit{-restricted}\\ \boldsymbol{F_{K,\mathfrak{p}}}\textit{-modules}\end{matrix}\right)\!\!\right)$$

Moreover, the following is true:

(a) $\mathrm{rk}_{F_{K,\mathfrak{p}}} D_{\mathfrak{p}}(V,\rho) = \dim_{F_{\mathfrak{p}}} V$ *for every rational $\mathfrak{p}$-adic Galois representation (V,ρ);*

(b) *the homomorphism $F_{K^{\mathrm{sep}},\mathfrak{p}} \otimes_{F_{\mathfrak{p}}} R_{\mathfrak{p}}(M,\tau) \to F_{K^{\mathrm{sep}},\mathfrak{p}} \otimes_{F_{K,\mathfrak{p}}} M$ is an isomorphism for every $\mathfrak{p}$-restricted $\boldsymbol{F_{K,\mathfrak{p}}}$-module (M,τ);*

(c) *the homomorphism $F_{K^{\mathrm{sep}},\mathfrak{p}} \otimes_{F_{K,\mathfrak{p}}} D_{\mathfrak{p}}(V,\rho) \to F_{K^{\mathrm{sep}},\mathfrak{p}} \otimes_{F_{\mathfrak{p}}} V$ is an isomorphism for every rational $\mathfrak{p}$-adic Galois representation (V,ρ).*

Proof. Proposition 4.4 implies this rational version. In fact, '$D_{\mathfrak{p}} = D'_{\mathfrak{p}}$ for rational $\mathfrak{p}$-adic Galois representations' in the sense that $D_{\mathfrak{p}}(V,\rho)$ coincides with $(A_{K^{\mathrm{sep}},\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} V)^{\Gamma_K}$, the definition of $D'_{\mathfrak{p}}$ applied to (V,ρ), and similarly $R_{\mathfrak{p}} = R'_{\mathfrak{p}}$ for $\mathfrak{p}$-restricted $\boldsymbol{F_{K,\mathfrak{p}}}$-modules. The detailed proof also uses the fact that every rational $\mathfrak{p}$-adic Galois representation has a Γ_K-invariant full $A_{\mathfrak{p}}$-lattice (whereas not every restricted $\boldsymbol{F_{K,\mathfrak{p}}}$-module is $\mathfrak{p}$-restricted). □

PROPOSITION 4.7. *For every maximal ideal $\mathfrak{p} \neq \ker \iota$ of A, the following diagram commutes.*

$$\begin{tikzcd} A\text{-Isomot}_K \arrow[r, "\mathrm{V}_{\mathfrak{p}}"] \arrow[d, "I"'] & \mathrm{Rep}_{F_{\mathfrak{p}}}(\Gamma_K) \\ \boldsymbol{F_K}\text{-Mod} \arrow[r, "\boldsymbol{F_{K,\mathfrak{p}}} \otimes_{\boldsymbol{F_K}} (-)"'] & \boldsymbol{F_{K,\mathfrak{p}}}\text{-Mod} \arrow[u, "\mathrm{R}_{\mathfrak{p}}", "\cong"'] \end{tikzcd}$$

Proof. This follows directly from the construction of the categories and functors involved. □

We end this section by applying the main result of § 3, hence proving the 'first half' of Theorem 1.1. Let $\mathfrak{p}$ be a maximal ideal of A, let $F_{\mathfrak{p}}$ denote the completion of F at $\mathfrak{p}$, and set $\boldsymbol{F_{\mathfrak{p},K}} := \mathrm{Frac}(F_{\mathfrak{p}} \otimes_{\mathbb{F}_q} K, \mathrm{id} \otimes \sigma_q)$. Note that we have inclusions $\boldsymbol{F_K} \subset \boldsymbol{F_{\mathfrak{p},K}} \subset \boldsymbol{F_{K,\mathfrak{p}}}$, and that the latter is an equality if and only if K is a finite field. We set $\boldsymbol{A_{\mathfrak{p},K}} := \boldsymbol{F_{\mathfrak{p},K}} \cap \boldsymbol{A_{K,\mathfrak{p}}}$, and say that an $\boldsymbol{F_{\mathfrak{p},K}}$-module is $\mathfrak{p}$-*restricted* if it is restricted with respect to the inclusion $\boldsymbol{A_{\mathfrak{p},K}} \subset \boldsymbol{F_{\mathfrak{p},K}}$.

By what we have already proven, Theorem 1.1 (the semisimplicity conjecture) follows by proving that bold scalar extension $\boldsymbol{F_{K,\mathfrak{p}}} \otimes_{\boldsymbol{F_K}} (-)$ restricted to $\mathfrak{p}$-restricted $\boldsymbol{F_K}$-modules is semisimple on objects. Since

$$\boldsymbol{F_{K,\mathfrak{p}}} \otimes_{\boldsymbol{F_K}} (-) = (\boldsymbol{F_{K,\mathfrak{p}}} \otimes_{\boldsymbol{F_{\mathfrak{p},K}}} (-)) \circ (\boldsymbol{F_{\mathfrak{p},K}} \otimes_{\boldsymbol{F_K}} (-)),$$

and being semisimple on objects is a transitive property, we may subdivide our task into two parts.

THEOREM 4.8. *Let $\mathfrak{p}$ be a maximal ideal of A. Assume that the number of roots of unity of K is finite. The restriction of the functor of bold scalar extension $\boldsymbol{F_{\mathfrak{p},K}} \otimes_{\boldsymbol{F_K}} (-)$ to restricted $\boldsymbol{F_K}$-modules is:*

(a) *$F_{\mathfrak{p}}/F$-fully faithful;*

(b) *semisimple on objects; and*

(c) *maps $\mathfrak{p}$-restricted modules to $\mathfrak{p}$-restricted modules.*

Proposition 4.9. *Every completion $F_\mathfrak{p}$ of F at a place $\mathfrak{p}$ is a separable field extension.*

Proof. Let us start with the special case of $F=\mathbb{F}_q(t)$ completed at $\mathfrak{p}=(t)$, so $F_\mathfrak{p}=\mathbb{F}_q(\!(t)\!)$. By [Bou58, V.§ 15.4] it is sufficient to prove the following: if $f_1,\ldots,f_m\in\mathbb{F}_q(\!(t)\!)$ are linearly independent over $\mathbb{F}_q(t)$, then so are the f_i^p. Without loss of generality, assume that $f_i\in\mathbb{F}_q[\![t]\!]$, and that for certain $g_i\in k[t]$ we have $\sum_i g_i f_i^p=0$. We must show that all g_i are zero.

Since $\mathbb{F}_q$ is perfect, we may write $g_i=:\sum_{j=0}^{p-1} g_{ij}^p t^j$ for certain $g_{ij}\in k[t]$. These defining equations, together with $\sum_i g_i f_i^p=0$, imply that for all j we have $\sum_i g_{ij}^p f_i^p=0$. By extracting pth roots of both sides we obtain $\sum_i g_{ij} f_i=0$ for all j. By assumption the f_i are linearly independent, so we have $g_{ij}=0$ for all i and j. Therefore, all g_i are zero, as required.

Let us return to the general setting. We choose a local parameter $t\in F$ at $\mathfrak{p}$. Denoting the residue field of F at $\mathfrak{p}$ by $\mathbb{F}_\mathfrak{p}$, we have $F_\mathfrak{p}=\mathbb{F}_\mathfrak{p}(\!(t)\!)$ and the following commutative diagram of inclusions.

$$\begin{tikzcd} \mathbb{F}_q(t) \arrow[r] \arrow[d] & F \arrow[d] \\ \mathbb{F}_q(\!(t)\!) \arrow[r] & \mathbb{F}_\mathfrak{p}(\!(t)\!) \end{tikzcd}$$

We have just seen that $\mathbb{F}_q(t)\subset\mathbb{F}_q(\!(t)\!)$ is separable; clearly, so is $\mathbb{F}_q(\!(t)\!)\subset\mathbb{F}_\mathfrak{p}(\!(t)\!)$, hence $\mathbb{F}_q(t)\subset F_\mathfrak{p}$ is separable. Moreover, $\mathbb{F}_q(t)\subset F$ is separable algebraic since t is a local parameter. This implies that $F\subset F_\mathfrak{p}$ is separable by [Bou58, V.§ 15]. $\square$

Proof of Theorem 4.8. Since $F_\mathfrak{p}/F$ is separable by Proposition 4.9, Theorem 3.11 implies parts (a) and (b) of Theorem 4.8. Part (c) follows from the fact that $A_{(\mathfrak{p}),K}=F_K \Cap A_{K,\mathfrak{p}}$. $\square$

5. Tamagawa–Fontaine theory

In this section, we complete the proof of Theorem 1.1 with the help of what we term 'Tamagawa–Fontaine theory', since the basic ideas and a sketch of the proofs are due to Tamagawa [Tam95] and have some formal analogy to Fontaine theory.

Let $F,\mathbb{F}_q,A,\mathfrak{p}$ be as before, let $K/\mathbb{F}_q$ be a $\boxed{\textit{finitely generated}}$ field and let K^{sep} denote a separable closure of K with associated absolute Galois group $\Gamma_K:=\mathrm{Gal}(K^{\mathrm{sep}}/K)$. Recall that we have constructed bold rings $\boldsymbol{A_{\mathfrak{p},K}}\subset\boldsymbol{F_{\mathfrak{p},K}}$ and $\boldsymbol{A_{K,\mathfrak{p}}}\subset\boldsymbol{F_{K,\mathfrak{p}}}$, and that we call $\boldsymbol{F_{\mathfrak{p},K}}$- and $\boldsymbol{F_{K,\mathfrak{p}}}$-modules $\mathfrak{p}$-*restricted* if they are restricted with respect to these inclusions. To any $\mathfrak{p}$-restricted $\boldsymbol{F_{\mathfrak{p},K}}$-module $\boldsymbol{M}$, we associate the rational $\mathfrak{p}$-adic Galois representation

$$\mathrm{V}_\mathfrak{p}(\boldsymbol{M}):=\mathrm{R}_\mathfrak{p}(\boldsymbol{F_{K,\mathfrak{p}}}\otimes_{\boldsymbol{F_{\mathfrak{p},K}}}\boldsymbol{M}).$$

Definition 5.1. Following [Tam95], we say that a rational $\mathfrak{p}$-adic Galois representation is *quasigeometric* if it is isomorphic to $\mathrm{V}_\mathfrak{p}(\boldsymbol{M})$ for some $\mathfrak{p}$-restricted $\boldsymbol{F_{\mathfrak{p},K}}$-module $\boldsymbol{M}$.

The theory consists of constructing a bold ring $\boldsymbol{B}\subset\boldsymbol{F_{K^{\mathrm{sep}},\mathfrak{p}}}$ and developing the properties of the associated functor[7]

$$\mathrm{C}_\mathfrak{p}:=((\boldsymbol{B}\otimes_{F_\mathfrak{p}}(-))^{\Gamma_K}:\mathrm{Rep}_{F_\mathfrak{p}}(\Gamma_K)\to\boldsymbol{F_{\mathfrak{p},K}}\text{-}\mathrm{Mod}^{\mathfrak{p}\text{-res}}.$$

This allows us to determine which rational $\mathfrak{p}$-adic Galois representations are quasigeometric (those for which $\mathrm{rk}_{F_{\mathfrak{p},K}}(\mathrm{C}_\mathfrak{p}(V,\rho))=\dim_{F_\mathfrak{p}}V$), and its properties imply that $\boldsymbol{F_{K,\mathfrak{p}}}\otimes_{\boldsymbol{F_{\mathfrak{p},K}}}(-)$,

[7] 'C' for coreflection.

restricted to $\mathfrak{p}$-restricted $\boldsymbol{F_{\mathfrak{p},K}}$-modules, is fully faithful and semisimple on objects. Thereby, the proof of Theorem 1.1 is complete.

We choose to postpone the construction of $\boldsymbol{B}$ to the next section (Definitions 6.6 and 6.10), and develop the properties of $\mathrm{C}_{\mathfrak{p}}$ using only the properties of $\boldsymbol{B}$ given in the following claim. These properties will also be established in the next section (Theorem 6.23).

CLAIM 5.2. Assume that $K/\mathbb{F}_q$ is finitely generated. There exists a ring $B \subset F_{K^{\mathrm{sep}},\mathfrak{p}}$ with the following properties:

(a) $\sigma_{F_{K^{\mathrm{sep}},\mathfrak{p}}}(B) \subset B$ and $B^{\sigma} = F_{\mathfrak{p}}$;

(b) $\Gamma_K(B) \subset B$ and $B^{\Gamma_K} = F_{\mathfrak{p},K}$;

(c) every $\mathfrak{p}$-restricted $\boldsymbol{F_{\mathfrak{p},K}}$-module $\boldsymbol{M}$ fulfills $\mathrm{V}_{\mathfrak{p}}(\boldsymbol{M}) \subset \boldsymbol{B} \otimes_{\boldsymbol{F_{\mathfrak{p},K}}} \boldsymbol{M}$.

Note that the existence of such a ring of periods is a matter of *construction*, since property (b) requires $\boldsymbol{B}$ to be 'small enough' (as $(F_{K^{\mathrm{sep}},\mathfrak{p}})^{\Gamma_K} = F_{K,\mathfrak{p}}$ strictly contains $F_{\mathfrak{p},K}$ if K is infinite), whereas property (c) requires $\boldsymbol{B}$ to be 'large enough' (as it must contain the Galois-invariant elements of $\boldsymbol{F_{K^{\mathrm{sep}},\mathfrak{p}}} \otimes_{\boldsymbol{F_{\mathfrak{p},K}}} \boldsymbol{M}$ for every $\mathfrak{p}$-restricted $\boldsymbol{F_{\mathfrak{p},K}}$-module $\boldsymbol{M}$).

This claim will be justified in Theorem 6.23. Until the end of the proof of Theorem 5.15, *we will assume that Claim 5.2 is true*. Note that there exists a smallest ring with the properties required in Claim 5.2, the intersection of the (non-empty) set of such rings. What follows does not depend on our choice of $\boldsymbol{B}$. However, we might as well choose this canonical smallest $\boldsymbol{B}$ in the following, so we do.

LEMMA 5.3. *Let $\boldsymbol{M} = (M, \tau)$ be a $\mathfrak{p}$-restricted $F_{\mathfrak{p},K}$-module. Then the natural comparison isomorphism $F_{K^{\mathrm{sep}},\mathfrak{p}} \otimes_{F_{\mathfrak{p}}} \mathrm{V}_{\mathfrak{p}}(\boldsymbol{M}) \to F_{K^{\mathrm{sep}},\mathfrak{p}} \otimes_{F_{\mathfrak{p},K}} M$ of Proposition 4.6(b) descends to a Γ_K-equivariant isomorphism of $\boldsymbol{B}$-modules*

$$c_{\boldsymbol{M}} : \boldsymbol{B} \otimes_{F_{\mathfrak{p}}} \mathrm{V}_{\mathfrak{p}}(\boldsymbol{M}) \longrightarrow \boldsymbol{B} \otimes_{\boldsymbol{F_{\mathfrak{p},K}}} \boldsymbol{M}.$$

Proof. Claim 5.2(b) implies that the given isomorphism descends to a Γ_K-equivariant homomorphism of $\boldsymbol{B}$-modules

$$c_{\boldsymbol{M}} : \boldsymbol{B} \otimes_{F_{\mathfrak{p}}} \mathrm{V}_{\mathfrak{p}}(\boldsymbol{M}) \longrightarrow \boldsymbol{B} \otimes_{\boldsymbol{F_{\mathfrak{p},K}}} \boldsymbol{M}.$$

Since both sides are free B-modules of finite rank, it suffices to show that the determinant of $c_{\boldsymbol{M}}$ is an isomorphism. Since $\mathrm{V}_{\mathfrak{p}}$ is a tensor functor, we have

$$\det(c_{\boldsymbol{M}}) = c_{\det(\boldsymbol{M})},$$

so we may reduce to the case where $\mathrm{rk}(\boldsymbol{M}) = 1$. In this case, choosing a basis for both $\mathrm{V}_{\mathfrak{p}}(\boldsymbol{M})$ and $\boldsymbol{M}$, we see that $c_{\boldsymbol{M}}$ is given by left multiplication by an element $c(\boldsymbol{M}) \in B$. Choosing the dual bases of $\mathrm{V}_{\mathfrak{p}}(\boldsymbol{M}^{\vee})$ and $\boldsymbol{M}^{\vee}$, analogously $c_{\boldsymbol{M}^{\vee}}$ is given by left multiplication by an element $c(\boldsymbol{M}^{\vee}) \in B$.

By Proposition 4.6(c), the element $c(\boldsymbol{M})$ is invertible in $F_{K^{\mathrm{sep}},\mathfrak{p}}$. By naturality, its inverse $c(\boldsymbol{M})^{-1}$ coincides with $c(\boldsymbol{M}^{\vee})$. Since both $c(\boldsymbol{M})$ and $c(\boldsymbol{M}^{\vee})$ lie in B, $c_{\boldsymbol{M}}$ is indeed an isomorphism. □

We continue to exploit the consequences of Claim 5.2.

THEOREM 5.4. *The functor $\mathrm{V}_{\mathfrak{p}}$ on $\mathfrak{p}$-restricted $F_{\mathfrak{p},K}$-modules is fully faithful.*

Proof. Consider two $\mathfrak{p}$-restricted $\boldsymbol{F}_{\mathfrak{p},\boldsymbol{K}}$-modules $\boldsymbol{M}, \boldsymbol{N}$. By Lemma 5.3 we have a τ- and Γ_K-equivariant natural isomorphism

$$\boldsymbol{B} \otimes \boldsymbol{M}^\vee \otimes \boldsymbol{N} \longrightarrow \boldsymbol{B} \otimes \mathrm{V}_\mathfrak{p}(\boldsymbol{M}^\vee \otimes \boldsymbol{N}) = \boldsymbol{B} \otimes \mathrm{V}_\mathfrak{p}(\boldsymbol{M})^\vee \otimes \mathrm{V}_\mathfrak{p}(\boldsymbol{N}),$$

which implies that

$$(\boldsymbol{B} \otimes \boldsymbol{M}^\vee \otimes \boldsymbol{N})^{\Gamma,\tau} \cong (\boldsymbol{B} \otimes \mathrm{V}_\mathfrak{p}(\boldsymbol{M})^\vee \otimes \mathrm{V}_\mathfrak{p}(\boldsymbol{N}))^{\tau,\Gamma}.$$

Since $\mathrm{Hom}(\boldsymbol{M}, \boldsymbol{N}) = (\boldsymbol{M}^\vee \otimes \boldsymbol{N})^\tau$ coincides with the domain of this isomorphism, and $\mathrm{Hom}(\mathrm{V}_\mathfrak{p}(\boldsymbol{M})^\vee, \mathrm{V}_\mathfrak{p}(\boldsymbol{N})) = (\mathrm{V}_\mathfrak{p}(\boldsymbol{M})^\vee \otimes \mathrm{V}_\mathfrak{p}(\boldsymbol{N}))^{\Gamma_K}$ coincides with its target, we see that $\mathrm{V}_\mathfrak{p}$ is indeed fully faithful. □

DEFINITION 5.5.

(a) Let (V, ρ) be a rational $\mathfrak{p}$-adic Galois representation. We set

$$\mathrm{C}_\mathfrak{p}(V, \rho) := (B \otimes_{F_\mathfrak{p}} V)^{\Gamma_K},$$

taking Galois-invariants with respect to the diagonal action. Note that the σ-linear endomorphism of B induces a σ-linear endomorphism τ of $\mathrm{C}_\mathfrak{p}(V, \rho)$.

(b) Set $\boldsymbol{B'} := \boldsymbol{B} \cap \boldsymbol{A}_{\boldsymbol{K}^{\mathrm{sep}},\mathfrak{p}}$. Let (T, ρ) be an integral $\mathfrak{p}$-adic Galois representation. We set

$$\mathrm{C}'_\mathfrak{p}(T, \rho) := (B' \otimes_{A_\mathfrak{p}} T)^{\Gamma_K},$$

taking Galois-invariants with respect to the diagonal action. Note that the σ-linear endomorphism of B' induces a σ-linear endomorphism τ of $\mathrm{C}'_\mathfrak{p}(T, \rho)$.

LEMMA 5.6. *For every $\mathfrak{p}$-restricted $\boldsymbol{F}_{\mathfrak{p},\boldsymbol{K}}$-module $\boldsymbol{M}$, the comparison isomorphism $c_{\boldsymbol{M}}$ of Lemma 5.3 induces an isomorphism of $\boldsymbol{F}_{\mathfrak{p},\boldsymbol{K}}$-modules*

$$\mathrm{C}_\mathfrak{p}(\mathrm{V}_\mathfrak{p}\, \boldsymbol{M}) \xrightarrow{\cong} \boldsymbol{M}.$$

Proof. Take Γ_K-invariants. □

PROPOSITION 5.7. (a) *The functor $\mathrm{C}_\mathfrak{p}$ is an exact $F_\mathfrak{p}$-linear tensor functor.*

(b) *The functor $\mathrm{C}'_\mathfrak{p}$ is an exact $A_\mathfrak{p}$-linear tensor functor.*

Proof. The functors $\mathrm{C}'_\mathfrak{p}$ and $\mathrm{C}_\mathfrak{p}$ are left exact linear functors by definition. Let us show that they are tensor functors. We deduce this from the fact that the functors $D'_\mathfrak{p}$ and $D_\mathfrak{p}$ of § 3 are such.

Let us do this for $\mathrm{C}_\mathfrak{p}$, mutatis mutandis the proof is the same for $C'_\mathfrak{p}$. Consider a rational $\mathfrak{p}$-adic Galois representation $\boldsymbol{V} = (V, \rho)$. We have $D_\mathfrak{p}(\boldsymbol{V}) = (F_{K^{\mathrm{sep}},\mathfrak{p}} \otimes_{F_\mathfrak{p}} V)^{\Gamma_K}$ and $\mathrm{C}_\mathfrak{p}(\boldsymbol{V}) = (B \otimes_{F_\mathfrak{p}} V)^{\Gamma_K}$. Therefore, calculating in $F_{K^{\mathrm{sep}},\mathfrak{p}} \otimes_{F_\mathfrak{p}} V$, we have $\mathrm{C}_\mathfrak{p}(\boldsymbol{V}) = (B \otimes_{F_\mathfrak{p}} V) \cap D_\mathfrak{p}(\boldsymbol{V})$.

Given another rational $\mathfrak{p}$-adic Galois representation $\boldsymbol{W}$, we may apply these remarks to $\boldsymbol{V}$, $\boldsymbol{W}$ and $\boldsymbol{V} \otimes_{F_\mathfrak{p}} \boldsymbol{W}$. In $F_{K^{\mathrm{sep}},\mathfrak{p}} \otimes_{F_\mathfrak{p}} V \otimes_{F_\mathfrak{p}} W$ we calculate

$$\begin{aligned}
\mathrm{C}_\mathfrak{p}(V \otimes_{F_\mathfrak{p}} W) &= (B \otimes_{F_\mathfrak{p}} V \otimes_{F_\mathfrak{p}} W) \cap D_{F_\mathfrak{p}}(V \otimes_{F_\mathfrak{p}} W) \\
&= ((B \otimes_{F_\mathfrak{p}} V) \otimes_B (B \otimes_{F_\mathfrak{p}} W)) \cap (D_{F_\mathfrak{p}}(V) \otimes_{F_{K,\mathfrak{p}}} D_{F_\mathfrak{p}}(W)) \\
&= ((B \otimes_{F_\mathfrak{p}} V) \cap D_{F_\mathfrak{p}}(V)) \otimes_{F_{\mathfrak{p},K}} ((B \otimes_{F_\mathfrak{p}} W) \cap D_{F_\mathfrak{p}}(W)) \\
&= \mathrm{C}_\mathfrak{p}(V) \otimes_{F_{\mathfrak{p},K}} \mathrm{C}_\mathfrak{p}(W).
\end{aligned}$$

Finally, the right exactness of $\mathrm{C}_\mathfrak{p}$ and $C'_\mathfrak{p}$ follows formally from what we have proven. Again, we do this only for $\mathrm{C}_\mathfrak{p}$, mutatis mutandis the proof is the same for $C'_\mathfrak{p}$. Since $\mathrm{C}_\mathfrak{p}$ is a tensor functor and $\boldsymbol{V}$ admits a dual $\boldsymbol{V}^\vee$, the $\boldsymbol{F_{\mathfrak{p},K}}$-module $\mathrm{C}_\mathfrak{p}(\boldsymbol{V})$ has a dual, namely $\mathrm{C}_\mathfrak{p}(\boldsymbol{V}^\vee)$. Therefore, if

$$\boldsymbol{V}' \to \boldsymbol{V} \to \boldsymbol{V}'' \to 0$$

is a right exact sequence of rational $\mathfrak{p}$-adic Galois representations, then its image under $\mathrm{C}_\mathfrak{p}$ coincides with the dual of the image of the left exact sequence $0 \to (\boldsymbol{V}'')^\vee \to \boldsymbol{V}^\vee \to (\boldsymbol{V}')^\vee$. Since $\mathrm{C}_\mathfrak{p}$ is left exact, the image of this left exact sequence is left exact. Since dualisation is exact, the image of our original right exact sequence is right exact, and we are done. □

Lemma 5.8.

(a) *The ring $F_{K,\mathfrak{p}}$ is a finite product of fields, each isomorphic to a field of Laurent series $K'(\!(t)\!)$ for some finite extension K'/K.*

(b) *The underlying $F_{K,\mathfrak{p}}$-module of every restricted $\boldsymbol{F_{K,\mathfrak{p}}}$-module is free.*

Proof. Let $t \in A$ denote a local parameter at $\mathfrak{p}$, and let $\mathbb{F}_\mathfrak{p}$ denote the residue field of $\mathfrak{p}$. By definition, $F_{K,\mathfrak{p}} = \mathrm{Frac}(A_{K,\mathfrak{p}})$, and we have

$$A_{K,\mathfrak{p}} = \varprojlim_n ((A/\mathfrak{p}^n) \otimes_{\mathbb{F}_q} K) = \varprojlim_n ((\mathbb{F}_\mathfrak{p}[t]/t^n) \otimes_{\mathbb{F}_q} K) = (\mathbb{F}_\mathfrak{p} \otimes_{\mathbb{F}_q} K)[\![t]\!].$$

As in the proof of Lemma 3.7(a), $\mathbb{F}_\mathfrak{p} \otimes_{\mathbb{F}_q} K \cong (K')^{\times s}$ for some finite field extension K'/K and integer $s \geqslant 1$. It follows that $F_{K,\mathfrak{p}} = K'(\!(t)\!)^{\times s}$ has the property stated in part (a). Part (b) follows as in the proof of Lemma 3.7(b). □

Lemma 5.9.

(a) *The module B' is a projective $A_{\mathfrak{p},K}$-module.*

(b) *The module B is a projective $F_{\mathfrak{p},K}$-module.*

Proof. By Lemma 3.7, $F_{\mathfrak{p},K} = Q_1 \times \cdots \times Q_s$ is a finite product of fields. Setting $B_i := Q_i \otimes_{F_{\mathfrak{p},K}} B$, we obtain a decomposition $B = B_1 \times \cdots \times B_s$. Since the Q_i are fields, the B_i are free Q_i-modules, so B is a projective $F_{\mathfrak{p},K}$-module.

To show that this implies that B' is a projective $A_{\mathfrak{p},K}$-module, we need some notation. Choose a local parameter $t \in F$ at $\mathfrak{p}$. We have $F_{\mathfrak{p},K} \subset F_{K,\mathfrak{p}}$, and the latter ring splits as $F_{K,\mathfrak{p}} = Q'_1 \times \cdots \times Q'_s$ with $Q'_i \cong K'((t))$ for a finite field extension $K' \supset K$ by Lemma 5.8. We may thus identify the fields Q_i with subfields of $Q'_i = K'(\!(t)\!)$, note that Q_i *contains* t.

Under this identification, setting $R_i := Q_i \cap K'[\![t]\!]$, we have $A_{\mathfrak{p},K} = R_1 \times \cdots \times R_s$.

The ring B is a subring of

$$F_{K^{\mathrm{sep}},\mathfrak{p}} \cong (\mathbb{F}_\mathfrak{p} \otimes_k K^{\mathrm{sep}})(\!(t)\!) = (\mathbb{F}_\mathfrak{p} \otimes_k K \otimes_K K^{\mathrm{sep}})(\!(t)\!) = (K' \otimes_K K^{\mathrm{sep}})(\!(t)\!)^{\times s},$$

with B_i contained in the ith copy of $(K' \otimes_K K^{\mathrm{sep}})(\!(t)\!)$. The ring B' splits as $B'_1 \times \cdots \times B'_s$, where $B'_i := B' \cap B_i$ is the ring consisting of those elements of B_i which, viewed as elements of the ith copy of $(K' \otimes_K K^{\mathrm{sep}})(\!(t)\!)$ in $F_{K^{\mathrm{sep}},\mathfrak{p}}$, are power series, that is, lie in $(K' \otimes_K K^{\mathrm{sep}})[\![t]\!]$.

Let us show that B'_i is a free R_i-module, which implies that B' is a projective $A_{\mathfrak{p},K}$-module. For this, we choose a Q_i-basis $\{b_{ij}\}_{j \in J_i}$ of B_i. Under the identifications given above, each b_{ij} corresponds to a Laurent series $\sum b_{ijn} t^n$ in $(K' \otimes_K K^{\mathrm{sep}})(\!(t)\!)$. Now $K' \otimes_K K^{\mathrm{sep}} \cong (K^{\mathrm{sep}})^{\times r}$ for some $r \geqslant 1$, whereby $1 \otimes 1$ corresponds to an element $(e_1, \ldots, e_\rho)$. By multiplying b_{ij} with a suitable element of the form $(e_1 t^{n(i,j,1)}, \ldots, e_\rho t^{n(i,j,r)})$, we may assume that $b_{ijn} = 0$ for $n < 0$ and that b_{ij0} is invertible in $K' \otimes_K K^{\mathrm{sep}}$. Under this assumption, one may check that $\{b_{ij}\}$ is indeed an R_i-basis of B'_i. □

LEMMA 5.10.

(a) *The natural homomorphism* $A_{K,\mathfrak{p}} \otimes_{A_{\mathfrak{p},K}} B' \longrightarrow A_{K^{\mathrm{sep}},\mathfrak{p}}$ *is injective.*

(b) *The natural homomorphism* $F_{K,\mathfrak{p}} \otimes_{F_{\mathfrak{p},K}} B \longrightarrow F_{K^{\mathrm{sep}},\mathfrak{p}}$ *is injective.*

Proof.

(a) We use the following facts from commutative algebra: given an ideal $I \subset R$ of a commutative ring R such that $\bigcap I^n = 0$, let $\widehat{R} := \varprojlim_n R/I^n$ denote the I-adic completion of R. If M is a projective R-module, then the natural homomorphism $\widehat{R} \otimes_R M \to \widehat{M} := \varprojlim_n M/(I^n \cdot M)$ is injective. If $M \to N$ is an injective homomorphism of R-modules, then the induced homomorphism $\widehat{M} \to \widehat{N}$ is injective.
The first of these facts is checked easily for free R-modules, and this implies the statement for projective R-modules by the additivity of source and target. The second fact is a consequence of the left exactness of $\varprojlim$.
By Lemma 5.9, we may apply this to $R = A_{\mathfrak{p},K}$, $I = \mathfrak{p}$, $M = B'$ and $N = F_{K^{\mathrm{sep}},\mathfrak{p}}$, and obtain the desired injectivity of

$$A_{K,\mathfrak{p}} \otimes_{A_{\mathfrak{p},K}} B' \to \widehat{B'} \to \widehat{F_{K^{\mathrm{sep}},\mathfrak{p}}} = F_{K^{\mathrm{sep}},\mathfrak{p}}.$$

(b) This follows from part (a) by inverting any local parameter $t \in F$ at $\mathfrak{p}$. $\square$

PROPOSITION 5.11. (a) *For every integral* $\mathfrak{p}$*-adic representation* $\boldsymbol{T}$*, the following natural map is injective:*

$$A_{K,\mathfrak{p}} \otimes_{A_{\mathfrak{p},K}} C'_{\mathfrak{p}}(\boldsymbol{T}) \longrightarrow D'_{\mathfrak{p}}(\boldsymbol{T}).$$

(b) *For every rational* $\mathfrak{p}$*-adic representation* $\boldsymbol{V}$*, the following natural map is injective:*

$$F_{K,\mathfrak{p}} \otimes_{F_{\mathfrak{p},K}} \mathrm{C}_{\mathfrak{p}}(\boldsymbol{V}) \longrightarrow D_{\mathfrak{p}}(\boldsymbol{V}).$$

Proof.

(a) We calculate

$$\begin{aligned} A_{K,\mathfrak{p}} \otimes_{A_{\mathfrak{p},K}} C'_{\mathfrak{p}}(\boldsymbol{T}) &= A_{K,\mathfrak{p}} \otimes_{A_{\mathfrak{p},K}} (B' \otimes_{A_{\mathfrak{p}}} T)^{\Gamma_K} \\ &= (A_{K,\mathfrak{p}} \otimes_{A_{\mathfrak{p},K}} B' \otimes_{A_{\mathfrak{p}}} T)^{\Gamma_K} \\ &\subset (A_{K^{\mathrm{sep}},\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} T)^{\Gamma_K} \text{ by Lemma 5.10(a)} \\ &= D'_{\mathfrak{p}}(\boldsymbol{T}). \end{aligned}$$

(b) We may repeat the calculation of part (a), using Lemma 5.10(b). $\square$

PROPOSITION 5.12.

(a) *The functor* $C'_{\mathfrak{p}}$ *has values in restricted* $\boldsymbol{A_{\mathfrak{p},K}}$*-modules.*

(b) *The functor* $\mathrm{C}_{\mathfrak{p}}$ *has values in* $\mathfrak{p}$*-restricted* $\boldsymbol{F_{\mathfrak{p},K}}$*-modules.*

(c) *For every rational* $\mathfrak{p}$*-adic Galois representation* $\boldsymbol{V} = (V, \rho)$ *we have* $\mathrm{rk}_{F_{\mathfrak{p},K}} \mathrm{C}_{\mathfrak{p}}(\boldsymbol{V}) \leqslant \dim_{F_{\mathfrak{p}}} V$.

Proof. For every rational representation $\boldsymbol{V}$ there exists an integral representation $\boldsymbol{T} = (T, \rho)$ such that $\boldsymbol{V} = F_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} \boldsymbol{T}$, and then $\mathrm{C}_{\mathfrak{p}}(\boldsymbol{V}) = \boldsymbol{F_{\mathfrak{p},K}} \otimes_{\boldsymbol{A_{\mathfrak{p},K}}} C'_{\mathfrak{p}}(\boldsymbol{T})$. Therefore, it suffices to show that $C'_{\mathfrak{p}}(\boldsymbol{T})$ is a restricted $\boldsymbol{A_{\mathfrak{p},K}}$-module of rank bounded above by $\mathrm{rk}_{A_{\mathfrak{p}}} T$.

By Proposition 5.11(a), $A_{K,\mathfrak{p}} \otimes_{A_{\mathfrak{p},K}} C'_{\mathfrak{p}}(\boldsymbol{T})$ is a submodule of $D'_{\mathfrak{p}}(\boldsymbol{T})$, which is a free $A_{K,\mathfrak{p}}$-module of rank $\mathrm{rk}_{A_{\mathfrak{p}}} T$. Therefore, $C'_{\mathfrak{p}}(\boldsymbol{T})$ is a finitely generated projective $A_{\mathfrak{p},K}$-module. Since $D'_{\mathfrak{p}}(\boldsymbol{T})$ has a bijective τ_{lin}, its submodule $A_{K,\mathfrak{p}} \otimes_{A_{\mathfrak{p},K}} C'_{\mathfrak{p}}(\boldsymbol{T})$ has an injective τ_{lin}, and therefore $C'_{\mathfrak{p}}(\boldsymbol{T})$ has an injective τ_{lin} as well. Since $\sigma_{A_{\mathfrak{p},K}}$ permutes the factors of $A_{\mathfrak{p},K} = F_{\mathfrak{p},K} \cap A_{\mathfrak{p},K} \cong (Q_1 \times \cdots \times Q_s) \cap A_{\mathfrak{p},K} = (Q_1 \cap A_{\mathfrak{p},K}) \times \cdots \times (Q_s \cap A_{\mathfrak{p},K})$ the injectivity of τ_{lin} implies that $C'_{\mathfrak{p}}(\boldsymbol{T})$ is free of *constant* rank $r := \mathrm{rk}_{A_{\mathfrak{p},K}} C'_{\mathfrak{p}}(\boldsymbol{T}) \leqslant \mathrm{rk}_{\boldsymbol{A_{K,\mathfrak{p}}}} \boldsymbol{T}$, as in the proof of Lemma 3.7(b).

It remains to show that the τ_{lin} of $C'_{\mathfrak{p}}(\boldsymbol{T})$ is bijective. Clearly, this is the case if and only if the τ_{lin} of the determinant of $C'_{\mathfrak{p}}(\boldsymbol{T})$ is bijective. By Proposition 5.7(a), $C'_{\mathfrak{p}}$ is a tensor functor, so we obtain an inclusion

$$A_{K,\mathfrak{p}} \otimes_{A_{\mathfrak{p},K}} C'_{\mathfrak{p}}\Big(\bigwedge_{A_{\mathfrak{p}}}^{r} \boldsymbol{T}\Big) \subset D'_{\mathfrak{p}}\Big(\bigwedge_{A_{\mathfrak{p}}}^{r} \boldsymbol{T}\Big),$$

where the right-hand side is a restricted $\boldsymbol{A_{K,\mathfrak{p}}}$-module of rank at least one. Tracing through the definitions, we see that the left-hand side is saturated in the right-hand side, i.e. the quotient is a projective $A_{K,\mathfrak{p}}$-module. An application of the Snake lemma shows that this implies that $A_{K,\mathfrak{p}} \otimes_{A_{\mathfrak{p},K}} C'_{\mathfrak{p}}(\Lambda^r \boldsymbol{T})$ has bijective τ_{lin}. Now the equality $A_{\mathfrak{p},K}^{\times} = A_{K,\mathfrak{p}}^{\times} \cap A_{\mathfrak{p},K}$ implies that $C'_{\mathfrak{p}}(\boldsymbol{T})$ itself has bijective τ_{lin}. □

Proposition 5.13. *Let $\boldsymbol{V} = (V, \rho)$ be a rational $\mathfrak{p}$-adic Galois representation:*

(a) *$\boldsymbol{V}$ is quasigeometric if and only if $\mathrm{rk}_{F_{\mathfrak{p},K}} \mathrm{C}_{\mathfrak{p}}(\boldsymbol{V}) = \mathrm{rk}_{F_{\mathfrak{p}}} V$;*

(b) *$\mathrm{V}_{\mathfrak{p}}(\mathrm{C}_{\mathfrak{p}}(\boldsymbol{V}))$ is the largest quasigeometric subrepresentation of $\boldsymbol{V}$;*

(c) *if $\boldsymbol{V}$ is quasigeometric, then so is every subquotient of $\boldsymbol{V}$.*

Proof.

(a) Assume that $\boldsymbol{V} \cong \mathrm{V}_{\mathfrak{p}}(\boldsymbol{M})$ is quasigeometric. By Lemma 5.6, $\mathrm{C}_{\mathfrak{p}}(\mathrm{V}_{\mathfrak{p}}(\boldsymbol{M})) \cong \boldsymbol{M}$. Therefore, using the fact that $\mathrm{V}_{\mathfrak{p}}$ preserves ranks, we have

$$\mathrm{rk}\, \mathrm{C}_{\mathfrak{p}}(\boldsymbol{V}) = \mathrm{rk}\, \mathrm{C}_{\mathfrak{p}}(\mathrm{V}_{\mathfrak{p}}(\boldsymbol{M})) = \mathrm{rk}(\boldsymbol{M}) = \mathrm{rk}\, \mathrm{V}_{\mathfrak{p}}(\boldsymbol{M}) = \mathrm{rk}\, \boldsymbol{V},$$

as claimed.
Assume that we have an equality of ranks. By Proposition 5.11(b), the natural homomorphism $\boldsymbol{F_{K,\mathfrak{p}}} \otimes_{\boldsymbol{F_{\mathfrak{p},K}}} \mathrm{C}_{\mathfrak{p}}(\boldsymbol{V}) \longrightarrow D_{\mathfrak{p}}(\boldsymbol{V})$ is injective. Since $D_{\mathfrak{p}}$ preserves ranks, both sides are free of equal finite rank over the semisimple commutative ring $F_{K,\mathfrak{p}}$. So the homomorphism is an isomorphism. We set $\boldsymbol{M} := \mathrm{C}_{\mathfrak{p}}(\boldsymbol{V})$, a $\mathfrak{p}$-restricted $\boldsymbol{F_{\mathfrak{p},K}}$-module by Proposition 5.12. Then the following isomorphisms shows that $\boldsymbol{V}$ is quasigeometric:

$$\boldsymbol{V} \cong R_{\mathfrak{p}}(D_{\mathfrak{p}}(\boldsymbol{V})) \cong R_{\mathfrak{p}}(\boldsymbol{F_{K,\mathfrak{p}}} \otimes_{\boldsymbol{F_{\mathfrak{p},K}}} \mathrm{C}_{\mathfrak{p}}(\boldsymbol{V})) = \mathrm{V}_{\mathfrak{p}}(\mathrm{C}_{\mathfrak{p}}(\boldsymbol{V})) = \mathrm{V}_{\mathfrak{p}}(\boldsymbol{M}).$$

(b) The representation $\mathrm{V}_{\mathfrak{p}}(\mathrm{C}_{\mathfrak{p}}\, \boldsymbol{V})$ is quasigeometric by Proposition 5.12(b). Proposition 5.11(b) and the exactness of $\mathrm{V}_{\mathfrak{p}}$ imply that $\mathrm{V}_{\mathfrak{p}}(\mathrm{C}_{\mathfrak{p}}\, \boldsymbol{V})$ is a subrepresentation of $\boldsymbol{V}$. Let us show that it contains every other quasigeometric subrepresentation $\mathrm{V}_{\mathfrak{p}}(\boldsymbol{M}') \cong \boldsymbol{V}' \subset \boldsymbol{V}$. By restricting the isomorphism $c_{\boldsymbol{M}'}$ of Lemma 5.3 to Γ_K-invariants, we have $\boldsymbol{M}' = \mathrm{C}_{\mathfrak{p}}(\mathrm{V}_{\mathfrak{p}}\, \boldsymbol{M}')$. So using the left-exactness of $\mathrm{C}_{\mathfrak{p}}$, we see that

$$\boldsymbol{M}' = \mathrm{C}_{\mathfrak{p}}(\mathrm{V}_{\mathfrak{p}}\, \boldsymbol{M}') = \mathrm{C}_{\mathfrak{p}}\, \boldsymbol{V}' \subset \mathrm{C}_{\mathfrak{p}}\, \boldsymbol{V}.$$

In turn, since $\mathrm{V}_{\mathfrak{p}}$ is exact, this shows that $\boldsymbol{V}' = \mathrm{V}_{\mathfrak{p}}(\boldsymbol{M}') \subset \mathrm{V}_{\mathfrak{p}}(\mathrm{C}_{\mathfrak{p}}\, \boldsymbol{V})$, as claimed.

(c) Let $0 \to \boldsymbol{V}' \to \boldsymbol{V} \to \boldsymbol{V}'' \to 0$ be an exact sequence of representations, and assume that $\boldsymbol{V}$ is quasigeometric. Consider the induced sequence

$$0 \longrightarrow \mathrm{C}_{\mathfrak{p}}\, \boldsymbol{V}' \longrightarrow \mathrm{C}_{\mathfrak{p}}\, \boldsymbol{V} \longrightarrow \mathrm{C}_{\mathfrak{p}}\, \boldsymbol{V}' \longrightarrow 0. \tag{5.14}$$

It is exact by Proposition 5.7. Applying the exact functor $\mathrm{V}_{\mathfrak{p}}$, we obtain an exact sequence

$$0 \longrightarrow \mathrm{V}_{\mathfrak{p}}\, \mathrm{C}_{\mathfrak{p}}\, \boldsymbol{V}' \longrightarrow \boldsymbol{V} \longrightarrow \mathrm{V}_{\mathfrak{p}}\, \mathrm{C}_{\mathfrak{p}}\, \boldsymbol{V}'' \longrightarrow 0,$$

where $\boldsymbol{V} = \mathrm{V}_{\mathfrak{p}}\, \mathrm{C}_{\mathfrak{p}}\, \boldsymbol{V}$ by part (b). Now

$$\mathrm{rk}\, \boldsymbol{V} = \mathrm{rk}\, \mathrm{V}_{\mathfrak{p}}\, \mathrm{C}_{\mathfrak{p}}\, \boldsymbol{V}' + \mathrm{rk}\, \mathrm{V}_{\mathfrak{p}}\, \mathrm{C}_{\mathfrak{p}}\, \boldsymbol{V}'' \leqslant \mathrm{rk}\, \boldsymbol{V}' + \mathrm{rk}\, \boldsymbol{V}'' = \mathrm{rk}\, \boldsymbol{V}$$

implies that $\mathrm{rk}\, \mathrm{V}_{\mathfrak{p}}\, \mathrm{C}_{\mathfrak{p}}\, \boldsymbol{V}' = \mathrm{rk}\, \boldsymbol{V}'$ and $\mathrm{rk}\, \mathrm{V}_{\mathfrak{p}}\, \mathrm{C}_{\mathfrak{p}}\, \boldsymbol{V}' = \mathrm{rk}\, \boldsymbol{V}'$, so $\boldsymbol{V}' = \mathrm{V}_{\mathfrak{p}}\, \mathrm{C}_{\mathfrak{p}}\, \boldsymbol{V}'$ and $\boldsymbol{V}'' = \mathrm{V}_{\mathfrak{p}}\, \mathrm{C}_{\mathfrak{p}}\, \boldsymbol{V}''$ are both quasigeometric by part (a). □

We collect our results in a categorical reformulation.

THEOREM 5.15.

(a) *The functor* $\mathrm{V}_{\mathfrak{p}} : \boldsymbol{F}_{\mathfrak{p},\boldsymbol{K}}\text{-}\mathrm{Mod}^{\mathfrak{p}\text{-res}} \to \mathrm{Rep}_{F_{\mathfrak{p}}}(\Gamma_K)$ *is an exact* $F_{\mathfrak{p}}$*-linear tensor functor which is fully faithful and semisimple on objects.*

(b) *The pair* $(\mathrm{V}_{\mathfrak{p}}, \mathrm{C}_{\mathfrak{p}})$ *is an adjoint pair of functors, that is, for every* $\mathfrak{p}$*-restricted* $\boldsymbol{F}_{\mathfrak{p},\boldsymbol{K}}$*-module* $\boldsymbol{M}$ *and rational* $\mathfrak{p}$*-adic Galois representation* $\boldsymbol{V}$ *there exists a natural isomorphism of* $F_{\mathfrak{p}}$*-vector spaces*

$$\mathrm{Hom}(\mathrm{V}_{\mathfrak{p}}(\boldsymbol{M}), \boldsymbol{V}) \longrightarrow \mathrm{Hom}(\boldsymbol{M}, \mathrm{C}_{\mathfrak{p}}(\boldsymbol{V})).$$

(c) *The unit* $\mathrm{id} \Rightarrow \mathrm{C}_{\mathfrak{p}} \circ \mathrm{V}_{\mathfrak{p}}$ *of this adjunction is an isomorphism (so* $\mathrm{C}_{\mathfrak{p}}$ *is a 'coreflection' of the 'inclusion'* $\mathrm{V}_{\mathfrak{p}}$*).*

(d) *The counit* $\mathrm{V}_{\mathfrak{p}} \circ \mathrm{C}_{\mathfrak{p}} \Rightarrow \mathrm{id}$ *of this adjunction is a monomorphism.*

Proof. (a) The functor $\mathrm{V}_{\mathfrak{p}} = \mathrm{R}_{\mathfrak{p}} \circ (\boldsymbol{F}_{\boldsymbol{K},\mathfrak{p}} \otimes_{\boldsymbol{F}_{\mathfrak{p},\boldsymbol{K}}} (-))$ is an exact $F_{\mathfrak{p}}$-linear tensor functor as a composition of such. It is fully faithful by Theorem 5.4. Proposition 5.13(c) implies that $\mathrm{V}_{\mathfrak{p}}$ maps simple objects to simple objects, so it is semisimple on objects.

(b) Let us construct the inverse of the adjunction isomorphism for a given $\boldsymbol{M}$ and $\boldsymbol{V}$. Since $\mathrm{V}_{\mathfrak{p}}$ is fully faithful, we have a natural isomorphism

$$\mathrm{V}_{\mathfrak{p}} : \mathrm{Hom}(\boldsymbol{M}, \mathrm{C}_{\mathfrak{p}}\, \boldsymbol{V}) \longrightarrow \mathrm{Hom}(\mathrm{V}_{\mathfrak{p}}\, \boldsymbol{M}, \mathrm{V}_{\mathfrak{p}}\, \mathrm{C}_{\mathfrak{p}}\, \boldsymbol{V})$$

One the other hand, every homomorphism $\mathrm{V}_{\mathfrak{p}}\, \boldsymbol{M} \to \boldsymbol{V}$ has a quasigeometric image by Proposition 5.13(c), which must lie in $\mathrm{V}_{\mathfrak{p}}\, \mathrm{C}_{\mathfrak{p}}\, \boldsymbol{V}$ by Proposition 5.13(b). Therefore, $\mathrm{Hom}(\mathrm{V}_{\mathfrak{p}}\, \boldsymbol{M}, \mathrm{V}_{\mathfrak{p}}\, \mathrm{C}_{\mathfrak{p}}\, \boldsymbol{V}) = \mathrm{Hom}(\mathrm{V}_{\mathfrak{p}}\, \boldsymbol{M}, \boldsymbol{V})$, and we are done.

(c), (d) Both of these items follow from Proposition 5.13. □

Proof of Theorem 1.1. By Proposition 4.7, the functor $\mathrm{V}_{\mathfrak{p}}$ on A-isomotives coincides with $\mathrm{R}_{\mathfrak{p}} \circ (F_{K,\mathfrak{p}} \otimes_{F_K} (-)) \circ I$. By Proposition 4.2, I is semisimple on objects and has $\mathfrak{p}$-restricted values. The functor $(F_{K,\mathfrak{p}} \otimes_{F_K} (-))$ is a composition of the functors $(F_{\mathfrak{p},K} \otimes_{F_K} (-))$ and $(F_{K,\mathfrak{p}} \otimes_{F_{\mathfrak{p},K}} (-))$. The former is semisimple on restricted $\boldsymbol{F}_{\boldsymbol{K}}$-modules and maps $\mathfrak{p}$-restricted modules to $\mathfrak{p}$-restricted $\boldsymbol{F}_{\mathfrak{p},\boldsymbol{K}}$-modules by Theorem 4.8(b) and (c), whereas the latter is semisimple on $\mathfrak{p}$-restricted $\boldsymbol{F}_{\mathfrak{p},\boldsymbol{K}}$-modules by Theorem 5.15(a). The functor $\mathrm{R}_{\mathfrak{p}}$ is semisimple on $\mathfrak{p}$-restricted $\boldsymbol{F}_{\boldsymbol{K},\mathfrak{p}}$-modules since it is an equivalence of categories. Therefore, $\mathrm{V}_{\mathfrak{p}}$ is semisimple on objects, being a composition of such functors. □

We end this section with a proof of the Tate conjecture for A-motives.

Proposition 5.16. *Let K be a field which is finitely generated over a finite field. Let $\mathfrak{p} \neq \ker \iota$ be a maximal ideal of A.*

(a) *Let M, N be A-motives of characteristic ι. The natural homomorphism $A_{\mathfrak{p}} \otimes_A \operatorname{Hom}(M, N) \to \operatorname{Hom}(\mathrm{T}_{\mathfrak{p}}\, M, \mathrm{T}_{\mathfrak{p}}\, N)$ is an isomorphism.*

(b) *Let X, Y be A-isomotives of characteristic ι. The natural homomorphism $F_{\mathfrak{p}} \otimes_F \operatorname{Hom}(X, Y) \to \operatorname{Hom}(\mathrm{V}_{\mathfrak{p}}\, M, \mathrm{V}_{\mathfrak{p}}\, N)$ is an isomorphism.*

Proof.

(b) By Proposition 4.7 we have $\mathrm{V}_{\mathfrak{p}} = \mathrm{R}_{\mathfrak{p}} \circ (F_{K,\mathfrak{p}} \otimes_{F_K} (-)) \circ I$. As in the proof of Theorem 1.1, this reduces the proof that $\mathrm{V}_{\mathfrak{p}}$ is $F_{\mathfrak{p}}/F$-fully faithful to Proposition 4.2, Theorem 4.8(a), (c) and Theorem 5.15(a).

(a) The image of the given homomorphism $A_{\mathfrak{p}} \otimes_A \operatorname{Hom}(M, N) \to \operatorname{Hom}(\mathrm{T}_{\mathfrak{p}}\, M, \mathrm{T}_{\mathfrak{p}}\, N)$ is saturated; this is well known. Therefore, part (b) implies part (a). $\square$

6. Constructing a ring of periods

We turn to the laborious task of constructing a ring B which fulfills Claim 5.2.

Recall that we assume that K is a *finitely generated* field extension of a finite field $\mathbb{F}_q$ with q elements. We identify K with the function field $\mathbb{F}_q(X)$ of a proper normal variety X over $\mathbb{F}_q$. For every finite Galois extension $K^{\mathrm{sep}} \supset L \supset K$, let X_L be the normalisation of X in L, this is a proper normal variety over L.

Let Σ_L be the set of prime (Weil) divisors of X_L. For every Galois tower

$$K^{\mathrm{sep}} \supset L' \supset L \supset K$$

we have a projection map $\mathrm{pr}_{L,L'} : \Sigma_{L'} \longrightarrow \Sigma_L$, so we may let

$$\Sigma^{\mathrm{sep}} := \varprojlim_{L \supset K} \Sigma_L$$

be the projective limit along the projections $\mathrm{pr}_{L',L}$. Given a Galois extension $L \supset K$, an element $x_L \in \Sigma_L$ and an element $x \in \Sigma^{\mathrm{sep}}$, we say that x *lies over* x_L if x_L is the Lth component of x.

For each $x = (x_L)_L \in \Sigma^{\mathrm{sep}}$, there is a unique associated valuation

$$v_x : K^{\mathrm{sep}} \longrightarrow \mathbb{Q} \cup \{\infty\}$$

extending the normalised valuation v_{x_K} of K associated to x_K. Explicitly, for $f \in K^{\mathrm{sep}}$ we may choose a finite Galois extension $K \subset L \subset K^{\mathrm{sep}}$ containing f, and set $v_x(f) := v_{x_L}(f)/e_{x_L}$, where v_{x_L} denotes the normalised valuation of L associated to x_L, and e_{x_L} is the index of $v_{x_L}(K^*)$ in $v_{x_L}(L^*)$.

Let F be a global field with field of constants $\mathbb{F}_q$, and fix a place $\mathfrak{p}$ of degree $d := \deg \mathfrak{p}$ of F with residue field $\mathbb{F}_{\mathfrak{p}}$. We wish to extend v_x to a function on $F_{K^{\mathrm{sep}},\mathfrak{p}}$. For calculational reasons, we choose a local parameter $t \in F$ at $\mathfrak{p}$ and obtain identifications $A_{K^{\mathrm{sep}},\mathfrak{p}} = (\mathbb{F}_{\mathfrak{p}} \otimes_k K^{\mathrm{sep}})[\![t]\!]$ and $F_{K^{\mathrm{sep}},\mathfrak{p}} = (\mathbb{F}_{\mathfrak{p}} \otimes_k K^{\mathrm{sep}})(\!(t)\!) = A_{K^{\mathrm{sep}},\mathfrak{p}}[t^{-1}]$. Recall that by Lemma 5.8 the homomorphism

$$(\mathbb{F}_{\mathfrak{p}} \otimes_k K^{\mathrm{sep}}, \mathrm{id} \otimes \sigma) \longrightarrow ((K^{\mathrm{sep}})^{\times d}, \sigma') \tag{6.1}$$

mapping $x \otimes y$ to $(x \cdot \sigma_q^i(y))_{i=0}^{d-1}$ is an isomorphism of bold rings, with

$$\sigma'(z_0, \dots, z_{d-1}) = (z_{d-1}^q, z_0^q, \dots, z_{d-2}^q)$$

for $(z_0, \ldots, z_{d-1}) \in (K^{\mathrm{sep}})^{\times d}$. We denote the action of σ' on $(K^{\mathrm{sep}})^{\times d}$ simply by σ. Writing an element $f \in F_{K^{\mathrm{sep}},\mathfrak{p}}$ as $f = \sum_{i \gg -\infty} f_i t^i$ with $f_i = (f_{ij})_j \in (K^{\mathrm{sep}})^{\times d}$, we set

$$v_x(f) := \inf_i \min_j v_x(f_{ij}).$$

Moreover, for all $m, n \geqslant 1$ and $\Delta = (\delta_{ij}) \in \mathrm{Mat}_{m\times n}(F_{K^{\mathrm{sep}},\mathfrak{p}})$ we set

$$v_x(\Delta) := \inf_{i,j} v_x(\delta_{ij}).$$

PROPOSITION 6.2. *For each $x \in \Sigma^{\mathrm{sep}}$ and all $m, n \geqslant 1$, the function*

$$v_x : \mathrm{Mat}_{m\times n}(F_{K^{\mathrm{sep}},\mathfrak{p}}) \longrightarrow \mathbb{R} \cup \{\pm\infty\}$$

is well defined and independent of the choices made. For $m = n = 1$ and all $f, g \in F_{K^{\mathrm{sep}},\mathfrak{p}}$ it has the following properties:

(a) $v_x(f + g) \geqslant \min\{v_x(f), v_x(g)\}$;

(b) $v_x(fg) \geqslant v_x(f) + v_x(g)$ *(using the convention $-\infty + \infty = -\infty$);*

(c) $v_x(\sigma(f)) = q \cdot v_x(f)$.

Proof. Since $v_x(\mathbb{F}_{\mathfrak{p}}^{\times}) = 0$, the choice of local parameter does not influence the definition of v_x. Now properties (a) and (b) follow from short calculations using the semicontinuity of infima, whereas property (c) follows from (6.1). □

Remark 6.3. Note that, in general, we do not have $v_x(fg) = v_x(f) + v_x(g)$.

PROPOSITION 6.4. *For all integers $m, n \geqslant 1$, matrices $\Delta \in \mathrm{Mat}_{n\times n}(F_{K^{\mathrm{sep}},\mathfrak{p}})$ and column vectors $F \in {F_{K^{\mathrm{sep}},\mathfrak{p}}}^{\oplus n}$ the equation $\sigma^m(F) = \Delta \cdot F$ implies the inequality*

$$v_x(F) \geqslant \frac{1}{q^m - 1} v_x(\Delta).$$

Proof. If $v_x(\Delta) = -\infty$, the inequality stated is tautological, so we assume that $C := v_x(\Delta) \neq -\infty$. By a matrix version of Proposition 6.2, the equation $\sigma^m(F) = \Delta F$ would imply that $q^m \cdot v_x(F) \geqslant C + v_x(F)$. If also $v_x(F) \neq \pm\infty$, this would imply the claim of this proposition. However, if $v_x(F) = -\infty$, there is a problem. The following proof deals with all cases at once.

Write $F = (f_i)$ and $\Delta = (\delta_{ij})$ with $f_i, \delta_{ij} \in F_{K^{\mathrm{sep}},\mathfrak{p}}$. Furthermore, write $f_i = \sum_r f_{ir} t^r$ and $\delta_{ij} = \sum_s h_{ijs} t^s$ for $f_{ir}, \delta_{ijs} \in \mathbb{F}_{\mathfrak{p}} \otimes_k K^{\mathrm{sep}}$. By multiplying the entire equation by a suitable power of t, we may assume that these coefficients are zero for $r, s < 0$. By assumption we have $v_x(\delta_{ijs}) \geqslant C$, and by definition we have $v_x(f_{ir}) \neq -\infty$.

The equation $\sigma^m(F) = \Delta \cdot F$ means $\sigma^m(f_i) = \sum_{j=1}^n \delta_{ij} f_j$ for all i, and gives

$$\sum_{r\geqslant 0} \sigma^m(f_{ir}) t^r = \sum_{j=1}^n \sum_{a\geqslant 0} \sum_{b\geqslant 0} \delta_{ija} f_{jb} t^{a+b} = \sum_{r\geqslant 0} \biggl(\sum_{j=1}^n \sum_{l=0}^r \delta_{ijl} f_{j,r-l} \biggr) t^r.$$

From this we see that

$$\sigma^m(f_{ir}) = \sum_{j=1}^n \sum_{l=0}^r \delta_{ijl} f_{j,r-l} \tag{6.5}$$

and must prove that $v_x(f_{ir}) \geqslant C/(q^m - 1)$. We perform induction on r.

If $r = 0$, then for all i we have $\sigma^m(f_{i0}) = \sum_{j=1}^n \delta_{ij0} f_{j0}$ which gives $q^m \cdot v_x(f_{i0}) \geqslant \min_{j=1}^n (C + v_x(f_{j0}))$. Choosing j such that the minimum is attained we obtain $q^m v_x(f_{j0}) \geqslant C + v_x(f_{j0})$ and hence $v_x(f_{j0}) \geqslant C/(q^m-1)$. So by the choice of j, for all i we may deduce that $v_x(f_{i0}) \geqslant v_x(f_{j0}) \geqslant C/(q^m-1)$.

For $r>0$, equation (6.5) gives $q^m v_x(f_{ir}) \geqslant \inf_{j\leqslant n, l\leqslant r}(C+v_x(f_{jl}))$, hence by the induction hypothesis for all $r'<r$

$$q^m v_x(f_{ir}) \geqslant \min\biggl(\frac{q^m}{q^m-1}C, \min_{j=1}^{n}(C+v_x(f_{jr}))\biggr).$$

If $q^m C/(q^m-1)$ is smaller, we obtain $v_x(f_{ir}) \geqslant C/(q^d-1)$ for all i as in the case $r=0$. Otherwise, choosing j such that the inner minimum is attained, we first obtain $v_x(f_{jr}) \geqslant C/(q^m-1)$ and then $v_x(f_{ir}) \geqslant C/(q^m-1)$ for all i, as in the case $r=0$. □

We now turn to the definition of our ring of periods.

Definition 6.6. Following [Tam95], we set:

(a) $B^+ := \left\{ f \in F_{K^{\mathrm{sep}},\mathfrak{p}} : \begin{array}{ll} v_x(f) \neq -\infty & \text{for all } x \in \Sigma^{\mathrm{sep}} \\ v_x(f) \geqslant 0 & \text{for almost all } x \in \Sigma^{\mathrm{sep}} \end{array} \right\}$, 'almost all' meaning that the set of exceptions has finite image in Σ_K;

(b) $S := \{s \in A^{\times}_{K^{\mathrm{sep}},\mathfrak{p}} : (\sigma(s)/s) \in F_{\mathfrak{p}} \otimes_k K\}$.

Lemma 6.7. *We have that B^+ is a Γ_K-stable ring.*

Proof. The fact that B^+ is Γ_K-stable follows directly from its definition. That B^+ is a ring (closed under finite sums and products) follows from Proposition 6.2: clearly, B^+ contains 1. For $f \in B^+$ let Σ_f denote the finite subset of those elements of Σ_K over which there lies an element $x \in \Sigma^{\mathrm{sep}}$ such that $v_x(f) < 0$.

Given two elements $f, g \in B^+$, for all $x \in \Sigma^{\mathrm{sep}}$ by Proposition 6.2(a) we have $v_x(f+g) \geqslant \min(v_x(f), v_x(g))$, which is not equal to $-\infty$, since this is such for both $v_x(f)$ and $v_x(g)$. For all x whose image in Σ_K does not lie in its the finite subset $\Sigma_f \cup \Sigma_g$ we even have $v_x(f+g) \geqslant 0$. Therefore, $f+g$ is an element of B^+.

A similar proof, using Proposition 6.2(b), shows that $f \cdot g$ is an element of B^+. All in all, B^+ is a ring. □

Lemma 6.8. *We have $(B^+)^{\Gamma_K} = F_{\mathfrak{p}} \otimes_k K$.*

Proof. We note that $(B^+)^{\Gamma_K} = B^+ \cap F_{K,\mathfrak{p}}$. So the desired equality $(B^+)^{\Gamma_K} = F_{\mathfrak{p}} \otimes_k K$ is an equality of subrings of $F_{K,\mathfrak{p}}$. By Lemma 5.8(a), we have $F_{K,\mathfrak{p}} = (K')^e(\!(t)\!)$ for a finite Galois extension K'/K (it is Galois since $\mathbb{F}_{\mathfrak{p}} \supset k$ is Galois and $\mathbb{F}_{\mathfrak{p}} \otimes_k K \cong (K')^e$). The inclusion $F_{K,\mathfrak{p}} \subset F_{K^{\mathrm{sep}},\mathfrak{p}}$ corresponds to a homomorphism $(K')^e(\!(t)\!) \hookrightarrow (K^{\mathrm{sep}})^d(\!(t)\!)$ mapping the ith component of the source to d/e components of the target, according to the d/e different K-linear embeddings of K' in K^{sep}. It follows that the image of this homomorphism lies in $(K')^d(\!(t)\!)$.

Given an element $f \in F_{K,\mathfrak{p}}$, we may write it as a Laurent series $\sum_i f_i t^i$, with coefficients $f_i = (f_{i1}, \ldots, f_{id}) \in (K')^d$. We let V_f denote the k-vector subspace of K' generated by the f_{ij}. Clearly, $F_{\mathfrak{p}} \otimes_k K$ consists of those elements of $F_{K,\mathfrak{p}}$ such that $\dim_k V_f$ is finite.

On the other hand, by definition $(B^+)^{\Gamma_K}$ consists of those elements of $F_{K,\mathfrak{p}}$ such that $v_{x'}(f) \neq -\infty$ for all $x' \in \Sigma_{K'}$ and $v_{x'}(f) \geqslant 0$ for all but a finite number of $x' \in \Sigma_{K'}$.

Now, if $f \in F_{K,\mathfrak{p}}$ is an element of $F_{\mathfrak{p}} \otimes_k K$, then $\dim_k V_f$ is finite, so the subset of $\Sigma_{K'}$ consisting of the poles of the (coefficients of the) elements of V_f is finite, so f is an element of B^+ by our above characterisation.

On the other hand, if $f \in F_{K,\mathfrak{p}}$ is an element of B^+, then we may choose a finite subset $\Sigma_0 \subset \Sigma_{K'}$ such that $v_{x'}(f) \geqslant 0$ for all $x' \notin \Sigma_0$. For $x' \in \Sigma_0$, we set $n(x') := -v_{x'}(f)$, which is finite by assumption. Let X' denote the proper normal variety over k corresponding to K'. Since X' is proper, the space of global sections of

$$\mathcal{O}_{X'}\left(\sum_{x' \in \Sigma_0} n(x')x'\right)$$

is finite-dimensional. Since it contains V_f, this implies that $f \in F_{\mathfrak{p}} \otimes_k K$ by our above characterisation. □

LEMMA 6.9. *The subset S is a Γ_K-stable multiplicative subset of B^+.*

Proof. The fact that S is a Γ_K-stable multiplicative subset of $F_{K^{\mathrm{sep}},\mathfrak{p}}$ follows directly from its definition.

Let us show that S is contained in B^+. For $s \in S$ choose $f \in F_{\mathfrak{p}} \otimes_k K$ such that $\sigma(s) = f \cdot s$, such an f exists by definition of S. By Lemma 6.8 and Proposition 6.4, $v_x(s) \neq -\infty$ for all $x \in \Sigma^{\mathrm{sep}}$, and there exists a finite subset Σ_0 of Σ_K such that $v_x(f) \geqslant 0$ for all $x \in \Sigma^{\mathrm{sep}}$ not lying over Σ_0.

For all $x \in \Sigma^{\mathrm{sep}}$, Proposition 6.4 shows that $v_x(s) \geqslant v_x(f)/(q-1)$. So s has the required properties that $v_x(s) \neq -\infty$ for all $x \in \Sigma^{\mathrm{sep}}$ and $v_x(s) \geqslant 0$ for all $x \in \Sigma^{\mathrm{sep}}$ not lying over Σ_0, since this is the case for f. □

DEFINITION 6.10. Following [Tam95], we let $B \subset F_{K^{\mathrm{sep}},\mathfrak{p}}$ be the ring obtained by inverting $S \subset B^+$, and set $\boldsymbol{B} = (B, \sigma)$, where σ is the given ring endomorphism of $\boldsymbol{F_{K^{\mathrm{sep}},\mathfrak{p}}}$.

LEMMA 6.11. *The ring $\boldsymbol{B}$ is a bold ring with ring of scalars $F_{\mathfrak{p}}$.*

Proof. The ring B is clearly σ-stable since B^+ and S are. Furthermore, since $F_{\mathfrak{p}} \subset B$ and $B^{\sigma} \subset F^{\sigma}_{K^{\mathrm{sep}},\mathfrak{p}} = F_{\mathfrak{p}}$, we have $B^{\sigma} = F_{\mathfrak{p}}$. □

We say that an element $f \in F_{K^{\mathrm{sep}},\mathfrak{p}}$ has *order* $n \in \mathbb{Z}$ if, writing f as $\sum f_i t^i \in (\mathbb{F}_{\mathfrak{p}} \otimes_k K^{\mathrm{sep}})(\!(t)\!)$ we have $n = \inf\{i : f_i \neq 0\}$. We say that an element $f \in F_{K^{\mathrm{sep}},\mathfrak{p}}$ of order n has *invertible leading coefficient* if f_n is invertible in $\mathbb{F}_{\mathfrak{p}} \otimes_k K^{\mathrm{sep}}$. If f has order zero, then we will denote by $f(0)$ the leading coefficient of f. Note that the invertible elements of $A_{K^{\mathrm{sep}},\mathfrak{p}}$ are precisely the elements of $F_{K^{\mathrm{sep}},\mathfrak{p}}$ of order zero with invertible leading coefficient.

Remark 6.12. Let us set $t_i := e_i \cdot t \in F_{K^{\mathrm{sep}},\mathfrak{p}}$, where e_i is the standard basis vector of the ith copy of K^{sep} in the product $(K^{\mathrm{sep}})^d$. Clearly, an element $f \in F_{K^{\mathrm{sep}},\mathfrak{p}}$ is invertible if and only if we can write

$$f = \left(\prod_{i=0}^{d-1} t_i^{n_i}\right) \cdot \widetilde{f},$$

for certain $n_i \in \mathbb{Z}$, where $\widetilde{f}$ is an element of $A^{\times}_{K^{\mathrm{sep}},\mathfrak{p}}$.

LEMMA 6.13. *Every element $f \in A^{\times}_{K^{\mathrm{sep}},\mathfrak{p}}$ may be written as $f = \sigma(s)/s$ for some other element $s \in A^{\times}_{K^{\mathrm{sep}},\mathfrak{p}}$.*

Proof. We write $f=\sum_{i\geqslant 0} f_i t^i$ and use the 'ansatz' $s=\sum_{j\geqslant 0} s_j t^j$. This gives

$$\sum_r \sigma(s_r)t^r=\sigma(s)=sf=\sum_{i,j} f_i s_j t^{i+j}=\sum_r\left(\sum_{i=0}^r f_i s_{r-i}\right)t^r.$$

We proceed by induction. For $r=0$, we must solve $\sigma(s_0)=f_0s_0$. We write $f_0=(f_{0,0},\ldots,f_{0,d-1})$ and $s_0=(s_{0,0},\ldots,s_{0,d-1})$ for $f_{0,i},\, s_{0,i}\in K^{\mathrm{sep}}$. Note that by assumption all $f_{0,i}\neq 0$. Since

$$\sigma(s_0)=(s_{0,d-1}^q, s_{0,0}^q, s_{0,1}^1,\ldots,s_{0,d-1}^q)$$

our equation $\sigma(s_0)=f_0s_0$ is equivalent to the system of equations

$$s_{0,i}^q=f_{0,i+1}s_{0,i+1},\quad i\in\mathbb{Z}/d\mathbb{Z}.$$

This means, for instance, that $s_{0,0}=s_{0,d-1}^q/f_{0,0}$ and $s_{0,d-1}=s_{0,d-2}^q/f_{0,d-1}$, which gives

$$s_{0,0}^q=\frac{s_{0,d-1}^q}{f_{0,0}}=\frac{(s_{0,d-2}^q/f_{0,d-1})^q}{f_{0,0}}.$$

Iterating this substitution, we obtain the equation

$$s_{0,0}^{q^d}-(f_{0,1}^{q^{d-1}}\cdot f_{0,2}^{q^{d-2}}\cdots f_{0,d-1}^q\cdot f_{0,0})s_{0,0}=0.$$

Since all of the $f_{0,i}\neq 0$, the constant $\phi:=f_{0,1}^{q^{d-1}}\cdot f_{0,2}^{q^{d-2}}\cdots f_{0,d-1}^q\cdot f_{0,0}$ is non-zero, so this is a separable equation for $s_{0,0}$ and hence has a non-trivial solution in K^{sep}. The $s_{0,i}$ for $i\neq 0$ are then determined by the assignments $s_{0,i}:=s_{0,i-1}^q/f_{0,i}$, they are non-trivial since $s_{0,0}$ and the $f_{0,i}$ are.

Let us consider the case $r>0$, and write $s_r=(s_{r,0},\ldots,s_{r,d-1})$ and $f_r=(f_{r,0},\ldots,f_{r,d-1})$. In this case, the equation $\sigma(s_r)=\sum_{i=0}^r f_i s_{r-i}$ that we must solve is equivalent to the system of equations

$$s_{r,i+1}^q=\sum_{j=0}^r f_{i,0}s_{r-i,0}=:f_{0,i}s_{r,i}+C_{r,i},$$

where the $C_{r,i}\in K^{\mathrm{sep}}$ are constants dependant only on f and the $s_{r'}$ for $r'<r$.

We may use the same type of replacement as before, and obtain an equation

$$s_{r,0}^{q^d}-\phi\cdot s_{r,0}=C_r$$

with $C_r\in K^{\mathrm{sep}}$ a constant determined by the $C_{r,i}$. Again, this is a separable equation for $s_{r,0}$, so there exists a solution in K^{sep}. The $s_{r,i}$ for $i\neq 0$ are then determined by the equations $s_{r,i}=(s_{r,i+1}^q-C_{r,i})/f_{0,i}$.

Finally, since we may choose the $s_{0,i}$ to be non-zero, our solution s is in fact invertible in $A_{K^{\mathrm{sep}},\mathfrak{p}}$. ☐

Proposition 6.14. *The ring B is a Γ_K-stable ring, and $B^{\Gamma_K}\supset F_{\mathfrak{p},K}$.*

Proof. The ring B is clearly Γ_K-stable, since B^+ and S both are. We have $B^{\Gamma_K}=B\cap F_{K,\mathfrak{p}}$.

Let us show that $F_{\mathfrak{p},K}\subset B$. Consider $g/f\in F_{\mathfrak{p},K}$ with $f,g\in F_{\mathfrak{p}}\otimes_k K$. By Remark 6.12, we may assume that f is in $A_{K^{\mathrm{sep}},\mathfrak{p}}^\times$. By Lemma 6.13 there exists an element $s\in S$ with $f=\sigma(s)/s$. It follows that $g/f=gs/\sigma(s)\in B$, since $gs\in B^+$ by Lemma 6.7 and $\sigma(s)\in S$. ☐

We turn to the inclusion $B^{\Gamma_K} \subset F_{\mathfrak{p},K}$, which is more difficult. Consider $b = b^+/s \in B^{\Gamma_K}$, with $b^+ \in B^+$ and $s \in S \subset A^\times_{K^{\mathrm{sep}},\mathfrak{p}}$. We set $f := \sigma^d(s)/s$, which is an element of $F_\mathfrak{p} \otimes_k K$, and for $N \geqslant 0$; following [Tam04], we set

$$a_N := b \cdot f(t^{q^d}) \cdot f(t^{q^{2d}}) \cdots f(t^{q^{Nd}}) \in F_{K,\mathfrak{p}}.$$

Remark 6.15. Our goal is to show that for N large enough the element a_N lies in B^+. By Lemma 6.9 this will imply that $a_N \in F_\mathfrak{p} \otimes_k K$, and in particular that $b \in B$.

Lemma 6.16. *There exists a finite set $\Sigma_0 \subset \Sigma_K$ such that for all $N \geqslant 0$ and all $x \in \Sigma^{\mathrm{sep}}$ not lying above Σ_N we have $v_x(a_N) \geqslant 0$.*

Proof. The idea is to use that b^+, s and f all lie in B^+, and then apply Proposition 6.2(b). In order to handle $1/s$, which is not necessarily an element of B^+, we need some modifications. Let $s(0)$ denote the leading coefficient of s, and set $\widetilde{s} := s/s(0)$. Clearly, $\widetilde{s}$ is an element of S with leading coefficient 1. Setting $\widetilde{f} := \sigma^d(\widetilde{s})/\widetilde{s}$, we have $\widetilde{f} \in F_\mathfrak{p} \otimes_k K$ and $f = \mu \cdot \widetilde{f}$ with $\mu := \sigma^d(s(0))/s(0)$ an invertible element of $\mathbb{F}_\mathfrak{p} \otimes_k K$. Now by definition and Proposition 6.2(b), we have

$$\begin{aligned} v_x(a_N) &= v_x\left(\frac{b^+}{s} \cdot f(t^{q^d}) \cdots f(t^{q^{Nd}})\right) \\ &= v_x\left(\frac{\mu^N}{s(0)} \cdot b^+ \cdot \frac{1}{\widetilde{s}} \cdot \widetilde{f}(t^{q^d}) \cdots \widetilde{f}(t^{q^{Nd}})\right) \\ &\geqslant N \cdot v_x(\mu) + v_x\left(\frac{1}{s(0)}\right) + v_x(b^+) + v_x\left(\frac{1}{\widetilde{s}}\right) + N \cdot v_x(\widetilde{f}). \end{aligned}$$

Since $E := \{\mu, 1/s(0), b^+, \widetilde{f}\}$ is a finite subset of B^+, the set Σ'_0 of those $x \in \Sigma^{\mathrm{sep}}$ for which there exists an $e \in E$ such that $v_x(e) < 0$ has finite image in Σ_K. Call this image Σ_0, and consider any $x \in \Sigma_0$. Proposition 6.4 implies that $v_x(\widetilde{s}) \geqslant v_x(\widetilde{f})/(q^d - 1) \geqslant 0$. Since $\widetilde{s}$ has leading coefficient 1, we may calculate $1/\widetilde{s}$ via the geometric series, and obtain $v_x(1/\widetilde{s}) \geqslant 0$, using Proposition 6.2. Therefore, $v_x(a_N)$ is bounded below by a finite sum of non-negative numbers, so $v_x(a_N) \geqslant 0$ for all x not lying above Σ_0. □

Lemma 6.17 (Following [Tam94b]). *Let $s \in A^\times_{K^{\mathrm{sep}},\mathfrak{p}}$, $x \in \Sigma^{\mathrm{sep}}$ and $N \geqslant 0$ fulfill:*

(a) $v_x(s) \geqslant 0$*; and*

(b) $v_x(s(0)) < q^N$.

Then, for every $a \in F_{K,\mathfrak{p}}$ we have an inequality

$$v_x(\sigma^N(a)) \geqslant \left\lfloor \frac{v_x(s \cdot \sigma^N(a))}{q^N} \right\rfloor \cdot q^N,$$

where for $x \in \mathbb{R}$ the term $\lfloor x \rfloor$ denotes the largest integer smaller than x.

Proof. We write $s = \sum_{i \geqslant 0} s_i t^i$ and $b := \sigma^N(a) = \sum_i b_i t^i$ with coefficients $s_i \in \mathbb{F}_\mathfrak{p} \otimes_k K^{\mathrm{sep}}$ and $b_i \in \mathbb{F}_\mathfrak{p} \otimes_k K$. We may assume that $b_i = 0$ for $i < 0$. By assumption, $v_x(s_i) \geqslant 0$ for all i, and $v_x(s_0) < q^N$. Note that since s_0 is invertible, the inequality $v_x(s_0 \cdot b_i) \geqslant v_x(s_0) + v_x(b_i)$ is in fact an equality.

We set $C := \lfloor v_x(sb)/q^N \rfloor \cdot q^N$, must prove that $v_x(b_i) \geqslant C$ for all i, and do this by induction on i.

For $i = 0$, we consider the inequality $v_x(s_0) + v_x(b_0) = v_x(s_0 b_0) \geqslant C$. It implies that, $v_x(b_0) \geqslant C - v_x(s_0) > C - q^N$. However, by assumption the value of $v_x(b_0)$ lies in $q^N \cdot \mathbb{Z} \cup \{\infty\}$, and there exists no integral multiple of q^N strictly greater than $C-q^N$ and less than C. Therefore, we have $v_x(b_0)\geqslant C$.

For $i>0$, we have $s_0b_i=(sb)_i-\sum_{j=1}^{i}s_jb_{i-j}$. By induction, we deduce that

$$\begin{aligned}v_x(s_0b_i)&=v_x\bigg((sb)_i-\sum_{j=1}^{i}s_jb_{i-j}\bigg)\\&\geqslant\min\bigg(v_x\left((sb)_i\right),\ \min_{1\leqslant j\leqslant i}(v_x(s_j)+v_x(b_{i-j}))\bigg)\\&\geqslant\min(C,\min(0+C))\geqslant C.\end{aligned}$$

So $v_x(b_i)\geqslant C-v_x(s_0)$, which implies that $v_x(b_i)\geqslant C$ as in the case $i=0$ since $v_x(b_i)$ is an integral multiple of q^N and $0\leqslant v_x(s_0)<q^N$. □

Lemma 6.18. *There exists an $N_0\geqslant 1$ such that for all $N\geqslant N_0$ and all $x\in\Sigma^{\mathrm{sep}}$ we have $v_x(a_N)\neq-\infty$.*

Proof. By Lemma 6.18, there exists a finite set $\Sigma_0\subset\Sigma_K$ such that $v_x(a_N)\geqslant 0>-\infty$ for all x not lying above Σ_0. Hence, it suffices to prove that, for one given $x_K\in\Sigma_K$, there exists an integer $N_0\geqslant 1$ such that for all $N\geqslant N_0$ and all x lying above x_K we have $v_x(a_N)\neq-\infty$. We fix such an $x_K\in\Sigma_0$.

Let π denote a local parameter of K at x_K. For all x over x_K, we have $v_x(s)\geqslant v_x(f)/(q^d-1)>-\infty$ by Proposition 6.4, so that $s=\pi^{-n}\widetilde{s}$ for some $n\geqslant 0$ and $\widetilde{s}\in S$ satisfying $v_x(s)\geqslant 0$. As a first substep, we wish to show that it is sufficient to deal with the case $s=\widetilde{s}$. This will make our calculations easier.

If $n>0$, then

$$\widetilde{f}:=\frac{\sigma^d(\widetilde{s})}{\widetilde{s}}=\frac{\sigma^d(\pi^n)}{\pi^n}\cdot\frac{\sigma(s)}{s}=\pi^{n(q^d-1)}f\in F_{\mathfrak{p}}\ \otimes_k K,$$

and by setting $\widetilde{b^+}:=\pi^n b^+\in B^+$, we obtain $b=\widetilde{b^+}/\widetilde{s}$, so that

$$\begin{aligned}\widetilde{a_N}&:=b\cdot\widetilde{f}(t^{q^d})\cdots\widetilde{f}(t^{q^{Nd}})\\&=b\cdot\pi^{n(q^d-1)}f(t^{q^d})\cdots\pi^{n(q^d-1)}f(t^{q^{Nd}})\\&=\pi^{Nn(q^d-1)}a_N.\end{aligned}$$

In particular, $v_x(a_N)\neq-\infty$ if and only if $v_x(\widetilde{a_N})\neq-\infty$, and we may assume in the following without loss of generality that the $s\in A^\times_{K^{\mathrm{sep}},\mathfrak{p}}$ we are given fulfills $v_x(s)\geqslant 0$.

We remark that for all $g\in F_{K^{\mathrm{sep}},\mathfrak{p}}$ and $i\geqslant 0$ we have the formula

$$\sigma^{id}(g(t^{q^{id}}))=g^{q^{id}},\tag{6.19}$$

in particular for our given $f\in F_{\mathfrak{p}}\ \otimes_k K$.

Second, note that from $b^+=bs$ and $\sigma^d(s)=sf$ we obtain $\sigma^d(b^+)=\sigma^d(b)\sigma^d(s)=\sigma^d(b)sf$, and by induction for $N\geqslant 1$

$$\sigma^{Nd}(b^+)=\sigma^{Nd}(b)s\cdot(f\cdot\sigma^d(f)\cdots\sigma^{(N-1)d}(f)).\tag{6.20}$$

Hence,

$$\begin{aligned}
\sigma^{Nd}(a_N)s &= \sigma^{Nd}(b\cdot f(t^{q^d})\cdots f(t^{q^{Nd}}))\cdot s\\
&= \sigma^N(b)s\cdot\sigma^{Nd}(f(t^{q^d})\cdots f(t^{q^{Nd}}))\\
&= \sigma^N(b^+)\cdot\frac{\sigma^{Nd}(f(t^{q^d})\cdots f(t^{q^{Nd}}))}{\sigma^{(N-1)d}(f)\cdots f}\quad\text{by equation (6.20)}\\
&= \sigma^N(b^+)\cdot\prod_{i=1}^{N}\sigma^{(N-i)d}\left(\frac{\sigma^{id}(f(t^{q^{id}}))}{f}\right)\\
&= \sigma^N(b^+)\cdot\prod_{i=1}^{N}\sigma^{(N-i)d}(f^{q^{id}-1})\quad\text{by (6.19)}\\
&=: \sigma^N(b^+)\cdot\phi,
\end{aligned}$$

with $\phi\in F_{\mathfrak{p}}\otimes_k K$, so it follows that $v_x(\sigma^N(a_N)s)\geqslant q^N v_x(b^+)+v_x(\phi)\neq-\infty$.

Now if N is large enough, namely, $q^N>v_x(s(0))$, then Lemma 6.17 shows that $q^N v_x(a_N)=v_x(\sigma^N(a_N))\neq-\infty$, so $v_x(a_N)\neq-\infty$ as required. □

Proposition 6.21. *The ring B fulfills $B^{\Gamma_K}=F_{\mathfrak{p},K}$.*

Proof. By Proposition 6.14 it suffices to show that $B^{\Gamma_K}\subset F_{\mathfrak{p},K}$. For $b\in B^{\Gamma_K}$ and $N\geqslant 0$, define a_N as before Remark 6.15. Lemmas 6.16 and 6.18 show that for N large enough, a_N is an element of B^+. By construction, it is an Γ_K-invariant, so Lemma 6.7 shows that $a_N\in F_{\mathfrak{p}}\otimes_k K$. By definition, this shows that

$$b=\frac{a_N}{f(t^{q^d})\cdot f(t^{q^{2d}})\cdots f(t^{q^{Nd}})}$$

is an element of $F_{\mathfrak{p},K}$, since both a_N and the denominator lie in $F_{\mathfrak{p}}\otimes_k K\subset F_{\mathfrak{p},K}$. □

So far, we have shown that $\boldsymbol{B}$ is a well-defined Γ_K-stable bold ring with scalar ring $F_{\mathfrak{p}}$ and $B^{\Gamma_K}=F_{K,\mathfrak{p}}$. It remains to prove that B has property (c) of Claim 5.2.

Lemma 6.22. *Let $\boldsymbol{M}$ be a $\mathfrak{p}$-restricted $\boldsymbol{F_{\mathfrak{p},K}}$-module. Then $\mathrm{V}_{\mathfrak{p}}(\boldsymbol{M})\subset\boldsymbol{B}\otimes_{\boldsymbol{F_{\mathfrak{p},K}}}\boldsymbol{M}$.*

Proof. We may assume, by choosing a basis, that $\boldsymbol{M}=(F_{\mathfrak{p},K}^{\oplus n},\tau)$ with $\tau(m)=\Delta\sigma(m)$ for some matrix $\Delta\in\mathrm{GL}_n(F_{\mathfrak{p},K})$ and all m.

Since $\mathrm{V}_{\mathfrak{p}}(\boldsymbol{M})=(\boldsymbol{F_{K^{\mathrm{sep}},\mathfrak{p}}}\otimes\boldsymbol{M})^\tau$, we have to prove that for all $m\in F_{K^{\mathrm{sep}},\mathfrak{p}}^{\oplus n}$ the equation $\Delta\cdot\sigma(m)=m$ implies that all entries of m lie in B.

Let us denote the inverse of Δ by $\Delta^{-1}=(g_{ij}/f_{ij})_{i,j}$, with $g_{ij}\in F_{\mathfrak{p}}\otimes_k K$ and $f_{ij}\in(F_{\mathfrak{p}}\otimes_k K)\cap A_{K^{\mathrm{sep}},\mathfrak{p}}^{\times}$. Setting $f:=\prod_{i,j}f_{ij}$, we see that $\Delta^{-1}=1/f\Delta'$ for some matrix Δ' with entries in $F_{\mathfrak{p}}\otimes_k K\subset B^+$.

By Lemma 6.13, we may write $f=\sigma(s)/s$ for some $s\in S$. For any element $m\in M$ write $m':=sm$. Now the equation $\tau(m)=m$ is equivalent to the equation $\sigma(m')=\Delta'\cdot m'$. By Proposition 6.4, this implies that m' has entries in B^+, so in particular $m=m'/s$ has entries in B, as claimed. □

Theorem 6.23. *The ring B fulfills Claim 5.2.*

Proof. By construction, B is a subring of $F_{K^{\mathrm{sep}},\mathfrak{p}}$. By Lemma 6.11, it fulfills Claim 5.2(a). By Propositions 6.14 and 6.21, it fulfills Claim 5.2(b). By Lemma 6.22, it also fulfills Claim 5.2(c). □

7. Algebraic monodromy groups

We recall the setup of Tannakian duality.

Definition 7.1.

(a) Let F be a field. A *pre-Tannakian category over* F is an F-linear rigid tensor category $\mathcal{T}$ such that all objects are of finite length, and for which the natural homomorphism $F \to \mathrm{End}_{\mathcal{T}}(\mathbf{1})$ is an isomorphism.

(b) Let $\mathcal{T}$ be a pre-Tannakian category, and consider an object X of $\mathcal{T}$. Then $(\!(X)\!)_{\otimes}$ denotes the smallest full abelian subcategory of $\mathcal{T}$ closed under tensor products and subquotients in $\mathcal{T}$.

(c) Let $\mathcal{T}$ be a pre-Tannakian category over F. Let F'/F be a field extension. A *fibre functor* on $\mathcal{T}$ is a faithful F-linear exact tensor functor $\omega: \mathcal{T} \to \mathrm{Vec}_{F'}$, where $\mathrm{Vec}_{F'}$ denotes the category of finite-dimensional F'-vector spaces. If $F' = F$, the fibre functor is called *neutral*.

(d) A *Tannakian category over* F is a pre-Tannakian category for which there exists a fibre functor over some field extension F'/F.

(e) Let $\mathcal{T}$ be a Tannakian category over F, consider a fibre functor ω of $\mathcal{T}$ over F'/F, and fix an object X of $\mathcal{T}$. The *algebraic monodromy group* of X with respect to ω is the functor

$$G_\omega(X): (\!(F'\text{-algebras})\!) \longrightarrow (\!(\text{groups})\!),$$

mapping an F'-algebra R' to the group of tensor automorphisms of the functor $R' \otimes_{F'} \omega(-)$ from $(\!(X)\!)_{\otimes}$ to R'-modules.

Proposition 7.2. *Let $\mathcal{T}$ be a Tannakian category over F, consider a fibre functor ω of $\mathcal{T}$ over F'/F, and fix an object X of $\mathcal{T}$. Then the algebraic monodromy group of X with respect to $\mathcal{T}$ is representable by an affine group scheme over F'.*

Proof ([Sta08, Theorem 3.1.7(a)]). This seems to be well known (to the experts). □

Let $F, \mathbb{F}_q, A, K, \iota$ be as in § 2, and choose a maximal ideal $\mathfrak{p} \neq \ker \iota$. In § 2, we have constructed the category $A\text{-Isomot}_K$ of A-isomotives over K. Using either the results of § 2, or the embedding I of Proposition 4.2, we see that it is a pre-Tannakian category. The category $\mathrm{Rep}_{F_\mathfrak{p}}(\Gamma_K)$ is a Tannakian category, since it fulfills the properties required by a pre-Tannakian category, and the forgetful functor $U: \mathrm{Rep}_{F_\mathfrak{p}}(\Gamma_K) \to \mathrm{Vec}_{F_\mathfrak{p}}$ is a fibre functor.

In § 2, we also constructed the functor

$$\mathrm{V}_\mathfrak{p} = R_\mathfrak{p} \circ (F_{K,\mathfrak{p}} \otimes_{F_{\mathfrak{p},K}} (-)) \circ (F_{\mathfrak{p},K} \otimes_{F_K} (-)) \circ I: A\text{-Isomot}_K \to \mathrm{Rep}_{F_\mathfrak{p}}(\Gamma_K),$$

associating to an A-isomotive its rational Tate module. It is faithful, F-linear and exact as a composition of such functors. Therefore, $A\text{-Isomot}_K$ is Tannakian, with fibre functor $U \circ \mathrm{V}_\mathfrak{p}$.

Given an A-isomotive $\boldsymbol{X}$, we set $G_\mathfrak{p}(\boldsymbol{X}) := G_{U \circ \mathrm{V}_\mathfrak{p}}(X)$, the *algebraic monodromy group* of X at $\mathfrak{p}$. On the other hand, we may consider $\Gamma_\mathfrak{p}(\boldsymbol{X})$, the image of $\Gamma_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$ in $\mathrm{Aut}_{F_\mathfrak{p}}(\mathrm{V}_\mathfrak{p}(\boldsymbol{X}))$. This might be called the $\mathfrak{p}$-*adic monodromy group* of $\boldsymbol{X}$, or rather $\mathrm{V}_\mathfrak{p}(\boldsymbol{X})$.

Proposition 7.3. *Let F' be a field, V be a finite-dimensional F'-vector space and consider a subgroup $\Gamma \subset \mathrm{GL}(V)(F')$ with associated algebraic group $G := \overline{\Gamma}^{\mathrm{Zar}} \subset \mathrm{GL}(V)$. The natural homomorphism $G \to G_U(V)$, with target the algebraic monodromy group of V as a representation of G with respect to the forgetful fibre functor $U: \mathrm{Rep}_{F'}(\Gamma) \to \mathrm{Vec}_{F'}$, is an isomorphism.*

Proof ([Sta08, Proposition 3.3.3(b)]). This seems to be well known (to the experts). □

It follows that $\Gamma_{\mathfrak{p}}(\boldsymbol{X})$ is a Zariski-dense subgroup of the group of $F_{\mathfrak{p}}$-rational points of the algebraic monodromy group of $\mathrm{V}_{\mathfrak{p}}(\boldsymbol{X})$ with respect to the forgetful fibre functor U of $\mathrm{Rep}_{F_{\mathfrak{p}}}(\Gamma_K)$. In order to prove Theorem 1.2(a), we must compare $G_{U\circ \mathrm{V}_{\mathfrak{p}}}(\boldsymbol{X})$ and $G_U(\mathrm{V}_{\mathfrak{p}}\,\boldsymbol{X})$. It is here that we invoke one of the main results of the author's previous article [Sta08].

THEOREM 7.4. *Let F'/F be a separable field extension, $\mathcal{T}$ a Tannakian category over F, $\mathcal{T}'$ a Tannakian category over F' and $\omega : \mathcal{T}' \to \mathrm{Vec}_{F'}$ a neutral fibre functor. Let $V : \mathcal{T} \to \mathcal{T}'$ be an F-linear exact functor which is F'/F-fully faithful, and semisimple on objects.*

For every object X of $\mathcal{T}$ the natural homomorphism $G_\omega(V(X)) \to G_{\omega\circ V}(X)$ is an isomorphism of algebraic groups.

Proof. See [Sta08, Proposition 3.1.8]. □

Proof of Theorem 1.2(a). Theorem 1.1 and Proposition 5.16 show that $\mathrm{V}_{\mathfrak{p}}$ has the properties required in Theorem 7.4. Together with Proposition 7.3, we see that the image of $\Gamma_{\mathfrak{p}}(\boldsymbol{X}) \to G_{\mathfrak{p}}(\boldsymbol{X})(F_{\mathfrak{p}})$ is indeed Zariski dense in $G_{\mathfrak{p}}(\boldsymbol{X})$ for every A-isomotive $\boldsymbol{X}$. □

DEFINITION 7.5. A semisimple F-algebra E is *separable* if the center of each simple factor of E is a separable field extension of F.

PROPOSITION 7.6. *Let F' be a field, V a finite-dimensional F'-vector space, and consider a closed algebraic subgroup $G \subset \mathrm{GL}(V)$. If V is semisimple as a representation of G, and $\mathrm{End}_G(V)$ is a separable F'-algebra, then the identity component G° is a reductive group.*

Proof ([Sta08, Proposition 3.2.1]). This seems to be well known (to the experts). □

Proof of Theorem 1.2(b). Let $\boldsymbol{X}$ be a semisimple A-isomotive with separable endomorphism algebra. By Theorem 1.2(a) the algebraic monodromy group $G := G_{\mathfrak{p}}(\boldsymbol{X})$ acts faithfully on $\mathrm{V}_{\mathfrak{p}}(\boldsymbol{X})$, the rational Tate module of $\boldsymbol{X}$. Since $\mathrm{V}_{\mathfrak{p}}$ is fully faithful by Proposition 5.16, $\mathrm{End}_G(\mathrm{V}_{\mathfrak{p}}\,\boldsymbol{X}) \cong F_{\mathfrak{p}} \otimes_F \mathrm{End}(\boldsymbol{X})$, so this is a semisimple separable $F_{\mathfrak{p}}$-algebra by [Bou58, no. 7, § 5, Proposition 6, Corollaire]. Therefore, Proposition 7.6 implies that G° is indeed a reductive group. □

ACKNOWLEDGEMENTS

This article as well as [Sta08] are developments of my Ph.D. thesis. It is my pleasure to thank Richard Pink for his guidance during my doctoral studies. I also wish to thank Akio Tamagawa for helpful email exchanges, and encouraging me to publish this article on my own. Finally, I thank the main referee for his helpful report, and Gebhard Böckle, Theo Bühler, Urs Hartl and both referees for their comments and suggestions concerning the presentation of this material.

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